Intuitionistic multi-agent subatomic natural deduction for belief and knowledge

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Introduction

The phenomena

1.1 If John believes that so-and-so, and Mary believes that so-and-so implies such-and-such, then both believe that such-and-such. (distributed belief)

1.2 If John knows that Mary knows that so-and-so, and if Mary knows that John knows that so-and-so then both know that so-and-so. (mutual knowledge)

1.3 If everyone knows that its not the case that so-and-so, then its not the case that John believes that so-and-so. (universal knowledge)

1.4 Hob thinks a witch has blighted Bob's mare, and Nob wonders whether she (the same witch) killed Cob's sow. (intentional identity)

Introduction

The ideal

A proof system which

- is suitable for the analysis of constructive reasoning with complex multi-agent belief (resp. knowledge) constructions
- has good proof-theoretic properties (normalization, subformula property)
- permits a proof-theoretic semantics for the intensional operators for intuitionistic belief and knowledge which explains their meaning entirely by appeal to the structure of derivations

Introduction

Not a viable method

Internalization:

1. Step:
   Translate possible worlds truth conditions for modal operators into introduction rules
2. Step:
   Obtain elimination rules by means of inversion principles
3. Step:
   Explain the meaning of the modal operators in terms of canonical derivations in the proof system generated via Step 1 and Step 2

Foundational problem:

Proof-theoretic semantics of modal operators is based on their model-theoretic semantics!

Proof-interpretation for belief

BHK-clause for belief (cf. [14])

IB. A proof of $\mathrm{B}(A)$ is given by presenting a proof of A.

We aim at a formal conception of proof. A proof may be closed or open.

Introduction

Familiar: Natural deduction

Deals with **superatomic** inference.

Less familiar: Subatomic natural deduction

Deals with **superatomic** and **subatomic** inference.

The proposal: Multi-agent subatomic natural deduction

Deals with **agent-relative** superatomic and subatomic inference.

Related work: Multi-agent natural deduction

- A. Cimatti and L. Serafini (1995): Multi-agent reasoning with belief contexts—the approach and a case study, LNCS.
- P. Piwek (2007): Meaning and dialogue coherence—a proof-theoretic investigation, JoLLI.

Motivating subatomic systems

A problem with atomic systems

Atomic systems (e.g., Prawitz, Troelstra & Schwichtenberg) undermine the subformula property:

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x Fx}{Fa}}{Gbc} \qquad \dfrac{}{Hd}}{Ie}}{\exists x Ix} \tag{1}$$

Subatomic systems avoid this problem.

Agent-relative subatomic systems

Agent-relative subatomic systems

An **agent-relative subatomic system** $\mathcal{S}_{\boldsymbol{a}}$ is a pair $\langle \mathcal{I}_{\boldsymbol{a}}, \mathcal{R}_{\boldsymbol{a}} \rangle$, where $\mathcal{I}_{\boldsymbol{a}}$ is an agent-relative subatomic base and $\mathcal{R}_{\boldsymbol{a}}$ is a set of agent-labelled I/E-rules for atomic sentences.

Agent-relative subatomic bases

An **agent-relative subatomic base** $\mathcal{I}_{\boldsymbol{a}}$ is a 3-tuple $\langle \mathcal{C}, \mathcal{P}, v_{\boldsymbol{a}} \rangle$, where $\mathcal{C}$ is the set of individual (or nominal) constants, $\mathcal{P}$ is the set of predicate constants, $\boldsymbol{a}$ is an *agent-label* (*agent*, for short), and $v_{\boldsymbol{a}}$ is such that:

1. For any $\alpha \in \mathcal{C}$, $v_{\boldsymbol{a}} : \mathcal{C} \to \wp(Atm)$, where $v_{\boldsymbol{a}}(\alpha) \subseteq Atm(\alpha)$.
2. For any $\varphi^n \in \mathcal{P}$, $v_{\boldsymbol{a}} : \mathcal{P} \to \wp(Atm)$, where $v_{\boldsymbol{a}}(\varphi^n) \subseteq Atm(\varphi^n)$.

For any $\tau \in \mathcal{C} \cup \mathcal{P}$, we define: $\tau\Gamma^{\boldsymbol{a}} =_{def} v_{\boldsymbol{a}}(\tau)$. $\tau\Gamma^{\boldsymbol{a}}$ is the set of **agent-relative term assumptions** for τ.

Agent-relative subatomic systems

Agent-labelled I/E-rules for atomic sentences

$\mathcal{R}_{\boldsymbol{a}}$ is a set of **agent-labelled I/E-rules for atomic sentences**:

$$\dfrac{\boldsymbol{a}\,\dfrac{\mathcal{D}_0}{\varphi_0^n\Gamma^{\boldsymbol{a}}} \quad \boldsymbol{a}\,\dfrac{\mathcal{D}_1}{\alpha_1\Gamma^{\boldsymbol{a}}} \quad \cdots \quad \boldsymbol{a}\,\dfrac{\mathcal{D}_n}{\alpha_n\Gamma^{\boldsymbol{a}}}}{\boldsymbol{a}\;\varphi_0^n\alpha_1\ldots\alpha_n}\,(as\mathrm{I})$$

where $\varphi_0^n\alpha_1\ldots\alpha_n \in \varphi_0^n\Gamma^{\boldsymbol{a}} \cap \alpha_1\Gamma^{\boldsymbol{a}} \cap \ldots \cap \alpha_n\Gamma^{\boldsymbol{a}}$

$$\dfrac{\boldsymbol{a}\,\dfrac{\mathcal{D}'}{\varphi_0^n\alpha_1\ldots\alpha_n}}{\boldsymbol{a}\;\tau_i\Gamma^{\boldsymbol{a}}}\,(as\mathrm{E}_i)$$

where $i \in \{0, \ldots, n\}$ and $\tau_i \in \{\varphi_0^n, \alpha_1, \ldots, \alpha_n\}$

Illustration: $\mathcal{S}_{\boldsymbol{a}}$-derivation

This derivation contains detours:

$$\boldsymbol{a}\,\dfrac{\chi^1\Gamma^{\boldsymbol{a}} \qquad \boldsymbol{a}\,\dfrac{\boldsymbol{a}\,\dfrac{\boldsymbol{a}\,\dfrac{\boldsymbol{a}\,\dfrac{\varphi^2\Gamma^{\boldsymbol{a}} \quad \alpha\Gamma^{\boldsymbol{a}} \quad \beta\Gamma^{\boldsymbol{a}}}{\varphi^2\alpha\beta}\,(as\mathsf{I})}{\varphi^2\Gamma^{\boldsymbol{a}}}\,(as\mathsf{E}_0) \qquad \boldsymbol{a}\,\dfrac{\langle\psi^2\gamma\delta\rangle_{\boldsymbol{a}}}{\delta\Gamma^{\boldsymbol{a}}}\,(as\mathsf{E}_2) \qquad \alpha\Gamma^{\boldsymbol{a}}}{\varphi^2\delta\alpha}\,(as\mathsf{I})}{\delta\Gamma^{\boldsymbol{a}}}\,(as\mathsf{E}_1)}{\chi^1\delta}\,(as\mathsf{I}) \qquad (2)$$

Agent-relative subatomic systems

Detour conversions for $as\mathrm{I}/\mathrm{E}_i$

$$\boldsymbol{a}\,\dfrac{\dfrac{\mathcal{D}_0}{\varphi_0^n\Gamma^{\boldsymbol{a}}} \quad \dfrac{\mathcal{D}_1}{\alpha_1\Gamma^{\boldsymbol{a}}} \quad \dots \quad \dfrac{\mathcal{D}_n}{\alpha_n\Gamma^{\boldsymbol{a}}}}{\boldsymbol{a}\,\dfrac{\varphi_0^n\alpha_1\dots\alpha_n}{\tau_i\Gamma^{\boldsymbol{a}}}\,(as\mathrm{E}_i)}\,(as\mathrm{I}) \quad \text{conv} \quad \dfrac{\mathcal{D}_i}{\tau_i\Gamma^{\boldsymbol{a}}}$$

(3) results from (2) by means of these conversions and is in normal form:

$$\boldsymbol{a}\,\dfrac{\chi^1\Gamma^{\boldsymbol{a}} \qquad \boldsymbol{a}\,\dfrac{\langle\psi^2\gamma\delta\rangle_{\boldsymbol{a}}}{\delta\Gamma^{\boldsymbol{a}}}\,(as\mathrm{E}_2)}{\chi^1\delta}\,(as\mathrm{I}) \tag{3}$$

Agent-relative subatomic systems

Rank

The **rank** $r(\varphi^n\alpha_1...\alpha_n)$ of an atomic sentence $\varphi^n\alpha_1...\alpha_n \in Atm$ is 1; this is also the rank of a maximum atomic sentence.

Cut rank

The **cut rank** $\mathrm{cr}(\mathcal{D})$ of an $\mathcal{S}_{\boldsymbol{a}}$-derivation $\mathcal{D}$ is a pair $\langle d, n\rangle$, where:

1. $d = max\{r(\varphi^n\alpha_1...\alpha_n) : \varphi^n\alpha_1...\alpha_n$ maximum atomic sentence in $\mathcal{D}\}$;
2. n is the number of maximum atomic sentences in $\mathcal{D}$.

Theorem (Normalization for $\mathcal{S}_{\boldsymbol{a}}$-systems)

Any derivation $\mathcal{D}$ in an $\mathcal{S}_{\boldsymbol{a}}$-system can be transformed into a normal $\mathcal{S}_{\boldsymbol{a}}$-derivation.

Digression: Language

Subexpression

1. Any formula A, predicate constant φ^n, nominal term o, and agent term $\underline{o}$ is a positive and strictly positive subexpression of itself.
2. If formula B is a subexpression of A, then so is any subformula of B.
3. Any predicate constant φ^n [nominal term o, agent term $\underline{o}$] occurring in formula A is a subexpression of A.

Agent-relative subatomic systems

$\mathcal{S}_{\boldsymbol{a}}$-Units

Let $\mathcal{D}$ be a derivation in an $\mathcal{S}_{\boldsymbol{a}}$-system.

1. An **$\mathcal{S}_{\boldsymbol{a}}$-unit** in $\mathcal{D}$ is either an occurrence of (i) an atomic sentence or (ii) an agent-relative term assumption $\tau\Gamma^{\boldsymbol{a}}$ in $\mathcal{D}$. We use $U_{\mathcal{S}_{\boldsymbol{a}}}, U'_{\mathcal{S}_{\boldsymbol{a}}}$ (possibly, with subscripts) for $\mathcal{S}_{\boldsymbol{a}}$-units.
2. In case $U_{\mathcal{S}_{\boldsymbol{a}}}$ is a term assumption $\tau\Gamma^{\boldsymbol{a}}$ in $\mathcal{D}$, τ is **the expression in** $U_{\mathcal{S}_{\boldsymbol{a}}}$.

Theorem (Subexpression property for $\mathcal{S}_{\boldsymbol{a}}$-systems)

If $\mathcal{D}$ is a normal $\mathcal{S}_{\boldsymbol{a}}$-derivation of an $\mathcal{S}_{\boldsymbol{a}}$-unit $U_{\mathcal{S}_{\boldsymbol{a}}}$ from a set of $\mathcal{S}_{\boldsymbol{a}}$-units Γ, then each $\mathcal{S}_{\boldsymbol{a}}$-unit in $\mathcal{D}$ is a subexpression of an expression in $\Gamma \cup \{U_{\mathcal{S}_{\boldsymbol{a}}}\}$.

Motivating subatomic identity systems

A problem with standard identity

Natural deduction systems with **standard I/E-rules for identity** undermine the subformula property:

$$\frac{b = c \qquad \dfrac{a = b \qquad Fa}{Fb}}{Fc} \tag{4}$$

Subatomic identity systems avoid this problem.

Non-primitive identity

Let φ^n be an n-ary predicate constant.

$$K^n_{\varphi^n}(o_1, o_2) =_{def}$$
$$\forall z_1 ... \forall z_{n-1} \forall z_n \; ((\varphi^n o_1 z_2 ... z_n \leftrightarrow \varphi^n o_2 z_2 ... z_n)$$
$$\& \; (\varphi^n z_1 o_1 ... z_n \leftrightarrow \varphi^n z_1 o_2 ... z_n)$$
$$\& \; ... \; \& \; (\varphi^n z_1 ... z_{n-1} o_1 \leftrightarrow \varphi^n z_1 ... z_{n-1} o_2))$$

Let $\varphi_1^{k_1}, ..., \varphi_m^{k_m}$ be all the atomic predicates in $\mathcal{P}$, where φ_i is k_i-ary.

$$o_1 \ddot{=} o_2 \quad =_{def} \quad K^{k_1}_{\varphi_1}(o_1, o_2) \; \& \; ... \; \& \; K^{k_m}_{\varphi_m}(o_1, o_2)$$

Agent-relative subatomic identity systems

An **agent-relative subatomic identity system** $\mathcal{S}_{\boldsymbol{a}}^{\ddot{=}}$ is a pair $\langle \mathcal{I}_{\boldsymbol{a}}, \mathcal{R}_{\boldsymbol{a}}^{\ddot{=}} \rangle$, where $\mathcal{R}_{\boldsymbol{a}}^{\ddot{=}}$ is $\mathcal{R}_{\boldsymbol{a}}$ extended with agent-labelled I/E-rules for $\ddot{=}$:

$$\boldsymbol{a}\,\dfrac{\dfrac{\overset{[\langle\varphi_1(\alpha_1)\rangle_{\boldsymbol{a}}]_{\boldsymbol{a}}^{(1_1)}}{\mathcal{D}_{1_1}}}{\varphi_1(\alpha_2)}\boldsymbol{a} \quad \dfrac{\overset{[\langle\varphi_1(\alpha_2)\rangle_{\boldsymbol{a}}]_{\boldsymbol{a}}^{(1_2)}}{\mathcal{D}_{1_2}}}{\varphi_1(\alpha_1)}\boldsymbol{a} \quad \ldots \quad \dfrac{\overset{[\langle\varphi_k(\alpha_1)\rangle_{\boldsymbol{a}}]_{\boldsymbol{a}}^{(k_1)}}{\mathcal{D}_{k_1}}}{\varphi_k(\alpha_2)}\boldsymbol{a} \quad \dfrac{\overset{[\langle\varphi_k(\alpha_2)\rangle_{\boldsymbol{a}}]_{\boldsymbol{a}}^{(k_2)}}{\mathcal{D}_{k_2}}}{\varphi_k(\alpha_1)}\boldsymbol{a}}{\alpha_1 \ddot{=} \alpha_2}\,(\ddot{=}\mathrm{I}),\ 1_1, \ldots, k_2$$

$$\boldsymbol{a}\,\dfrac{\boldsymbol{a}\,\dfrac{\mathcal{D}_1}{\alpha_1 \ddot{=} \alpha_2} \quad \boldsymbol{a}\,\dfrac{\mathcal{D}_{i_2}}{\varphi_i(\alpha_1)}}{\varphi_i(\alpha_2)}\,(\ddot{=}\mathrm{E}_i 1) \qquad \boldsymbol{a}\,\dfrac{\boldsymbol{a}\,\dfrac{\mathcal{D}_1}{\alpha_1 \ddot{=} \alpha_2} \quad \boldsymbol{a}\,\dfrac{\mathcal{D}_{i_1}}{\varphi_i(\alpha_2)}}{\varphi_i(\alpha_1)}\,(\ddot{=}\mathrm{E}_i 2)$$

where $i \in \{1, \ldots, k\}$

Agent-relative subatomic identity systems

Illustration: $\mathcal{S}_a^{\ddot{=}}$-derivations

$$
a\,\dfrac{a\,\dfrac{a\,\dfrac{\langle \alpha_1 \ddot{=} \alpha_2 \rangle_a \qquad \langle \varphi_i \alpha_1 \rangle_a}{\varphi_i \alpha_2}\,(\ddot{=}\mathrm{E}_i 1)}{\varphi_i \Gamma^a}\,(as\mathrm{E}_0) \qquad\qquad \alpha_3 \Gamma^a}{\varphi_i \alpha_3}\,(as\mathrm{I}) \tag{5}
$$

$$
a\,\dfrac{a\,\dfrac{a\,\dfrac{a\,\dfrac{[\langle \varphi_1^n(\alpha)\rangle_a]_a^{(1_1)}}{\varphi_1^n \Gamma} \qquad a\,\dfrac{[\langle \varphi_1^n(\alpha)\rangle_a]_a^{(1_1)}}{\alpha \Gamma}}{\varphi_1^n(\alpha)} \qquad \ldots \qquad a\,\dfrac{a\,\dfrac{[\langle \varphi_k^n(\alpha)\rangle_a]_a^{(k_2)}}{\varphi_k^n \Gamma} \qquad a\,\dfrac{[\langle \varphi_k^n(\alpha)\rangle_a]_a^{(k_2)}}{\alpha \Gamma}}{\varphi_k^n(\alpha)}}{\alpha \ddot{=} \alpha}\,(\ddot{=}\mathrm{I}),\, 1_1, \ldots, k_2 \qquad \langle \varphi_i^n(\alpha)\rangle_a}{\varphi_i^n(\alpha)}\,(\ddot{=}\mathrm{E}_i 2) \tag{6}
$$

Detour conversions for $\ddot{=}\mathsf{I}/\mathsf{E}_i j$

$$
\cfrac{
a\ \cfrac{
\begin{array}{c}[\langle\varphi_1^n(\alpha_1)\rangle_{\boldsymbol a}]^{(1_1)}\\ \mathcal{D}_{1_1}\\ \boldsymbol{a}\ \text{—}\\ \varphi_1^n(\alpha_2)\end{array}
\quad
\begin{array}{c}[\langle\varphi_1^n(\alpha_2)\rangle_{\boldsymbol a}]^{(1_2)}\\ \mathcal{D}_{1_2}\\ \boldsymbol{a}\ \text{—}\\ \varphi_1^n(\alpha_1)\end{array}
\quad \ldots \quad
\begin{array}{c}[\langle\varphi_k^n(\alpha_1)\rangle_{\boldsymbol a}]^{(k_1)}\\ \mathcal{D}_{k_1}\\ \boldsymbol{a}\ \text{—}\\ \varphi_k^n(\alpha_2)\end{array}
\quad
\begin{array}{c}[\langle\varphi_k^n(\alpha_2)\rangle_{\boldsymbol a}]^{(k_2)}\\ \mathcal{D}_{k_2}\\ \boldsymbol{a}\ \text{—}\\ \varphi_k^n(\alpha_1)\end{array}
}{\alpha_1 \ddot{=} \alpha_2}\ (\ddot{=}\mathsf{I})
\qquad
\boldsymbol{a}\ \cfrac{\mathcal{D}_{i_2}}{\varphi_i^n(\alpha_1)}
}{\varphi_i^n(\alpha_2)}\ (\ddot{=}\mathsf{E}_i 1)
$$

conv

$$
\boldsymbol{a}\ \cfrac{\mathcal{D}_{i_2}}{[\varphi_i^n(\alpha_1)]}
$$

$$
\boldsymbol{a}\ \cfrac{\mathcal{D}_{i_1}}{\varphi_i^n(\alpha_2)}
$$

The conversion with $\ddot{=}\mathsf{E}_i 2$ is similar.

Agent-relative subatomic identity systems

Rank

The **rank** $r(\alpha_1 \ddot{=} \alpha_2)$ of an identity sentence $\alpha_1 \ddot{=} \alpha_2$ (where possibly $\alpha_1 \equiv \alpha_2$) is 1; this is also the rank of a maximum identity sentence.

Cut rank

The **cut rank** $\mathrm{cr}(\mathcal{D})$ of an $\mathcal{S}^{\ddot{=}}$-derivation $\mathcal{D}$ is a 4-tuple $\langle d, n, e, m \rangle$, where:

1. d, n are like above;
2. $e = max\{r(\alpha_1 \ddot{=} \alpha_2) : \alpha_1 \ddot{=} \alpha_2$ maximum identity sentence in $\mathcal{D}\}$;
3. m is the number of maximum identity sentences in $\mathcal{D}$.

Theorem (Normalization for $\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}$-systems)

Any derivation $\mathcal{D}$ in an $\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}$-system can be transformed into a normal $\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}$-derivation.

Agent-relative subatomic identity systems

$\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}$-Units

Let $\mathcal{D}$ be a derivation in an $\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}$-system.

1. An $\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}$-**unit** in $\mathcal{D}$ is either an occurrence of (i) an atomic sentence, (ii) an identity sentence, or of (iii) an agent-relative term assumption $\tau\Gamma^{\boldsymbol{a}}$ in $\mathcal{D}$. We use $U_{\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}}, U'_{\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}}$ (possibly, with subscripts) for $\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}$-units.
2. In case $U_{\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}}$ is a term assumption $\tau\Gamma^{\boldsymbol{a}}$ in $\mathcal{D}$, τ is **the expression in** $U_{\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}}$.

$\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}$-Tracks

A **track** of an $\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}$-derivation $\mathcal{D}$ is a sequence of unit occurrences $U_{\mathcal{S}^{\ddot{=}}_{a0}}, \ldots, U_{\mathcal{S}^{\ddot{=}}_{an}}$ such that

1. $U_{\mathcal{S}^{\ddot{=}}_{a0}}$ is a top unit occurrence (i.e., a leaf) in $\mathcal{D}$;
2. $U_{\mathcal{S}^{\ddot{=}}_{ai}}$ for $i < n$ is not the minor premiss of an instance of $\ddot{=}\mathrm{E}_i\mathrm{j}$;
3. $U_{\mathcal{S}^{\ddot{=}}_{an}}$ is either (i) the minor premiss of an instance of $\ddot{=}\mathrm{E}_i\mathrm{j}$ or (ii) the conclusion of $\mathcal{D}$.

Agent-relative subatomic identity systems

Theorem (Subexpression property for $\mathcal{S}_{\boldsymbol{a}}^{\ddot{=}}$-systems)

If $\mathcal{D}$ is a normal $\mathcal{S}_{\boldsymbol{a}}^{\ddot{=}}$-derivation of an $\mathcal{S}_{\boldsymbol{a}}^{\ddot{=}}$-unit $U_{\mathcal{S}_{\boldsymbol{a}}^{\ddot{=}}}$ from a set of $\mathcal{S}_{\boldsymbol{a}}^{\ddot{=}}$-units Γ, then each $\mathcal{S}_{\boldsymbol{a}}^{\ddot{=}}$-unit in $\mathcal{D}$ is a subexpression of an expression in $\Gamma \cup \{U_{\mathcal{S}_{\boldsymbol{a}}^{\ddot{=}}}\}$.

NB: Below, $\varphi_i\alpha_2$ is a subexpression of a leaf.

$$\boldsymbol{a}\,\cfrac{\boldsymbol{a}\,\cfrac{\boldsymbol{a}\,\cfrac{\langle\alpha_1\ddot{=}\alpha_2\rangle_{\boldsymbol{a}} \quad \langle\varphi_i\alpha_1\rangle_{\boldsymbol{a}}}{\varphi_i\alpha_2}\,(\ddot{=}\mathrm{E}_i1)}{\varphi_i\Gamma^{\boldsymbol{a}}}\,(as\mathrm{E}_0) \qquad \alpha_3\Gamma^{\boldsymbol{a}}}{\varphi_i\alpha_3}\,(as\mathrm{I}) \tag{7}$$

Multi-agent subatomic natural deduction

Multi-agent belief systems (IBK-systems)

An *intuitionsitic multi-agent subatomic natural deduction belief system* $\mathcal{B}_{\mathbf{I}(\mathcal{S}^{\doteq}_{\mathcal{A}})}$ (abbr. **IBK**-system) is a pair $\langle \mathcal{I}_{\mathcal{A}}, \mathcal{R}_{\mathcal{A}} \rangle$, where

- $\mathcal{I}_{\mathcal{A}}$ is a multi-agent belief base and
- $\mathcal{R}_{\mathcal{A}}$ is a set of agent-labelled rules.

Multi-agent subatomic natural deduction

Multi-agent belief bases

Let $\mathcal{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ be a finite set of agents, let $\mathcal{S}^{\ddot{=}}_{\mathcal{A}} = \{\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}_1}, \ldots, \mathcal{S}^{\ddot{=}}_{\boldsymbol{a}_n}\}$, and let $\underline{\mathcal{C}} = \{\underline{a}_1, \ldots, \underline{a}_n\}$ be the set of agent constants.

A **multi-agent belief base** $\mathcal{I}_{\mathcal{A}}$ is a tuple $\langle \mathcal{A}, \mathcal{S}^{\ddot{=}}_{\mathcal{A}}, \underline{\mathcal{C}}, f, g, h \rangle$, where for each $i \in \{1, \ldots, n\}$:

$$
\begin{array}{lll}
f : \mathcal{A} \to \mathcal{S}^{\ddot{=}}_{\mathcal{A}} & \text{such that} & f(\boldsymbol{a}_i) = \mathcal{S}^{\ddot{=}}_{\boldsymbol{a}_i}, \\
g : \mathcal{A} \to \underline{\mathcal{C}} & \text{such that} & g(\boldsymbol{a}_i) = \underline{a}_i, \text{ and} \\
h : \underline{\mathcal{C}} \to \mathcal{C}_i & \text{such that} & h(g(\boldsymbol{a}_i)) = \alpha_i, \text{ where } \mathcal{C}_i \in f(\boldsymbol{a}_i).
\end{array}
$$

Agent-labelled logical rules

$\mathcal{R}_{\mathcal{A}}$ is a set which contains, for each $\boldsymbol{a} \in \mathcal{A}$ with (possibly primed) $\boldsymbol{a}$-subscripts $i, j, k, l, m \in \{1, \ldots, n\}$, the following **agent-labelled rules**:

Multi-agent subatomic natural deduction

Agent-labelled rules: Connectives and absurdity

$$\dfrac{\mathbf{a}_j \dfrac{\mathcal{D}_1}{A} \qquad \mathbf{a}_k \dfrac{\mathcal{D}_2}{B}}{A \& B} \, \mathbf{a}_i \; (\&\mathrm{I}) \qquad \mathbf{a}_j \dfrac{\mathcal{D}_1}{A \& B} \atop \mathbf{a}_i \dfrac{}{A} \; (\&\mathrm{E1}) \qquad \mathbf{a}_j \dfrac{\mathcal{D}_1}{A \& B} \atop \mathbf{a}_i \dfrac{}{B} \; (\&\mathrm{E2})$$

$$\mathbf{a}_i \dfrac{\mathbf{a}_j \dfrac{\mathcal{D}_1}{A}}{A \vee B} \; (\vee\mathrm{I1}) \qquad \mathbf{a}_i \dfrac{\mathbf{a}_j \dfrac{\mathcal{D}_1}{B}}{A \vee B} \; (\vee\mathrm{I2}) \qquad \mathbf{a}_i \dfrac{\mathbf{a}_j \dfrac{\mathcal{D}_1}{A \vee B} \qquad \mathbf{a}_k \dfrac{[\langle A \rangle_{\mathbf{a}_l}]^{(u)}_{\mathbf{a}_i} \atop \mathcal{D}_2}{C} \qquad \mathbf{a}_{k'} \dfrac{[\langle B \rangle_{\mathbf{a}_{l'}}]^{(v)}_{\mathbf{a}_i} \atop \mathcal{D}_3}{C}}{C} \; (\vee\mathrm{E}),\ u, v$$

$$\mathbf{a}_i \dfrac{\mathbf{a}_j \dfrac{[\langle A \rangle_{\mathbf{a}_k}]^{(u)}_{\mathbf{a}_i} \atop \mathcal{D}_1}{B}}{A \supset B} \; (\supset\mathrm{I}),\ u \qquad \mathbf{a}_i \dfrac{\mathbf{a}_j \dfrac{\mathcal{D}_1}{A \supset B} \qquad \mathbf{a}_k \dfrac{\mathcal{D}_2}{A}}{B} \; (\supset\mathrm{E}) \qquad \mathbf{a}_i \dfrac{\mathbf{a}_j \dfrac{\mathcal{D}_1}{\bot}}{A} \; (\bot\mathrm{i})$$

In $\bot$i: $A \in Atm$.

Multi-agent subatomic natural deduction

Agent-labelled rules: Universal quantifier

$$\dfrac{\boldsymbol{a}_i \; \dfrac{\boldsymbol{a}_j \; \mathcal{D}_1}{A(x/o)}}{\forall x A}\,(\forall \mathrm{I}) \qquad \dfrac{\boldsymbol{a}_i \; \dfrac{\boldsymbol{a}_j \; \mathcal{D}_1}{\forall x A}}{A(x/o)}\,(\forall \mathrm{E})$$

Side conditions:

1. In $\forall$I: (i) if o is a proper variable y, then $o \equiv x$ or o is not free in A, and o is not free in any assumption of a formula which is open in the derivation of $A(x/o)$; (ii) if o is a nominal constant, then o does neither occur in an undischarged assumption of a formula, nor in $\forall x A$, nor in a term assumption leaf $o\Gamma^{\boldsymbol{a}_k}$; (iii) o is a nominal constant and $\boldsymbol{a}_j \; \dfrac{\mathcal{D}_1}{A(x/o)}$ for all $o \in \mathcal{C}$.
2. In $\forall$E: o is free for x in A.

We write $\forall$I.i, $\forall$I.ii, $\forall$I.iii when we use the rule $\forall$I according to the conditions given in (i), (ii), and (iii).

Multi-agent subatomic natural deduction

Agent-labelled rules: Existential quantifier

$$\dfrac{\mathbf{a}_j \ \dfrac{\mathcal{D}_1}{A(x/o)}}{\mathbf{a}_i \ \exists x A}\,(\exists\mathrm{I}) \qquad \dfrac{\mathbf{a}_j \ \dfrac{\mathcal{D}_1}{\exists x A} \qquad \begin{array}{c}[\langle A(x/o)\rangle_{\mathbf{a}_l}]^{(u)}_{\mathbf{a}_i} \\ \mathbf{a}_k \ \dfrac{\mathcal{D}_2}{C}\end{array}}{\mathbf{a}_i \ C}\,(\exists\mathrm{E}),\ u$$

Side conditions:

1. In ∃E: (i) if o is a proper variable y, then $o \equiv x$ or o is not free in A, and o is not free in C nor in any assumption of a formula which is open in the derivation of the upper occurrence of C other than $[\langle A(x/o)\rangle_{\mathbf{a}_l}]^{(u)}_{\mathbf{a}_i}$; (ii) if o is a nominal constant, then o does neither occur in an undischarged assumption of a formula, nor in $\exists x A$, nor in C, nor in a term assumption leaf $o\Gamma^{\mathbf{a}_m}$.
2. In ∃I: o is free for x in A.

We write ∃E.i, ∃E.ii when we use the rule ∃E according to the conditions given in (i) and (ii).

Multi-agent subatomic natural deduction

Agent-labelled rules: Belief and knowledge

$$\dfrac{\boldsymbol{a}_k \; \dfrac{\mathcal{D}_1}{A}}{\boldsymbol{a}_i \; \mathsf{B}_{\underline{a}_j}(A)} \; (\mathsf{B}_{\underline{a}_j}\mathsf{I}) \qquad \dfrac{\boldsymbol{a}_k \; \dfrac{\mathcal{D}_1}{\mathsf{B}_{\underline{a}_j}(A)}}{\boldsymbol{a}_i \; A} \; (\mathsf{B}_{\underline{a}_j}\mathsf{E})$$

$$\dfrac{\boldsymbol{a}_k \; \dfrac{\mathcal{D}_1}{A}}{\boldsymbol{a}_i \; \mathsf{K}_{\underline{a}_j}(A)} \; (\mathsf{K}_{\underline{a}_j}\mathsf{I}) \qquad \dfrac{\boldsymbol{a}_k \; \dfrac{\mathcal{D}_1}{\mathsf{K}_{\underline{a}_j}(A)}}{\boldsymbol{a}_i \; A} \; (\mathsf{K}_{\underline{a}_j}\mathsf{E})$$

Side condition on $\mathsf{K}_{\underline{a}_j}\mathsf{I}$:
A does neither depend on a term assumption nor on an open formula assumption.

Observation: I/E-rules for knowledge do not expand.

Multi-agent subatomic natural deduction

I-rule for belief: Kinds of belief

$\mathcal{D}_1$ may contain the following kinds of leaves: discharged formula assumptions (DFA), undischarged formula assumptions (UFA), and term assumptions (TA).

Category	$\mathcal{D}_1$ contains UFA	$\mathcal{D}_1$ contains DFA	$\mathcal{D}_1$ contains TA
C1	yes	yes	yes
C2	yes	yes	no
C3	yes	no	yes
C4	yes	no	no
C5	no	yes	yes
C6	no	yes	no
C7	no	no	yes
C8	no	no	no

We may distinguish the following **kinds of belief**: *conditional belief* (C1-4), *unconditional belief* (C5-8), *purely hypothetical belief* (C4), *knowledge* (C6), *purely basic belief* (C7), *empty belief* (C8).

Multi-agent subatomic natural deduction

I-rule for belief: Interactivity parameters

The following combinations of **interactivity parameters** (abbr. IP) are possible with respect to the rules for $B_{\underline{a}}, K_{\underline{a}}$, where

IP1 = agent label of premiss
IP2 = agent label of conclusion
IP3 = subscripted agent constant

	IP1	IP2	IP3
IP1		distinct	distinct
IP2	same		distinct
IP3	same	same	

Multi-agent subatomic natural deduction

Agent-labelled rules: Agent quantifiers

Let $\mathrm{E} \in \{\mathrm{B}, \mathrm{K}\}$:

$$\dfrac{\boldsymbol{a}_i \; \dfrac{\boldsymbol{a}_j \; \mathcal{D}_1}{\mathrm{E}_{(\underline{o}/\underline{a}_k)}(A)}}{\underline{\forall}\ \underline{x}\mathrm{E}_{\underline{x}}(A)} \; (\underline{\forall}\mathrm{I}) \qquad \dfrac{\boldsymbol{a}_i \; \dfrac{\boldsymbol{a}_k \; \mathcal{D}_1}{\underline{\forall}\ \underline{x}\mathrm{E}_{\underline{x}}(A)}}{\mathrm{E}_{(\underline{x}/\underline{a}_j)}(A)} \; (\underline{\forall}\mathrm{E})$$

Side condition on $\underline{\forall}$I: $\mathcal{D}_1$ derives $\mathrm{E}_{(\underline{o}/\underline{a}_k)}(A)$ for each $\underline{a}_k \in \underline{\mathcal{C}}$.

$$\dfrac{\boldsymbol{a}_i \; \dfrac{\boldsymbol{a}_j \; \mathcal{D}_1}{\mathrm{E}_{(\underline{x}/\underline{a}_j)}(A)}}{\underline{\exists}\ \underline{x}\mathrm{E}_{\underline{x}}(A)} \; (\underline{\exists}\mathrm{I}) \qquad \dfrac{\boldsymbol{a}_i \; \dfrac{\boldsymbol{a}_j \; \mathcal{D}_1}{\underline{\exists}\ \underline{x}\mathrm{E}_{\underline{x}}(A)} \qquad \begin{matrix}[\langle \mathrm{E}_{(\underline{o}/\underline{a}_m)}(A)\rangle_{\boldsymbol{a}_l}]^{(u)}_{\boldsymbol{a}_i} \\ \boldsymbol{a}_k \; \dfrac{\mathcal{D}_2}{C}\end{matrix}}{C} \; (\underline{\exists}\mathrm{E}),\ u$$

Side condition on $\underline{\exists}$E: $\underline{a}_m$ does neither occur in an undischarged assumption, nor in $\underline{\exists}\ \underline{x}\mathrm{E}_{\underline{x}}A$, nor in C.

Multi-agent subatomic natural deduction

Agent identity

Let A be an atomic formula.

$$\underline{K}^n_A(\underline{o}_1, \underline{o}_2) =_{def}$$

$$\underline{\forall}\ \underline{z}_1 \ldots \underline{\forall}\ \underline{z}_{n-1} \underline{\forall}\ \underline{z}_n\ ((B_{\underline{o}_1} B_{\underline{z}_2} \ldots B_{\underline{z}_n} A \leftrightarrow B_{\underline{o}_2} B_{\underline{z}_2} \ldots B_{\underline{z}_n} A)$$
$$\&\ (B_{\underline{z}_1} B_{\underline{o}_1} \ldots B_{\underline{z}_n} A \leftrightarrow B_{\underline{z}_1} B_{\underline{o}_2} \ldots B_{\underline{z}_n} A)$$
$$\&\ \ldots\ \&\ (B_{\underline{z}_1} \ldots B_{\underline{z}_{n-1}} \ldots B_{\underline{o}_1} A \leftrightarrow B_{\underline{z}_1} \ldots B_{\underline{z}_{n-1}} \ldots B_{\underline{o}_2} A))$$

Let $A_1, \ldots, A_m$ be a finite list of atomic formulae.

$$\underline{o}_1 \ \underline{\ddot{=}}\ \underline{o}_2 \quad =_{def} \quad \underline{K}_{A_1}(\underline{o}_1, \underline{o}_2)\ \&\ \ldots\ \&\ \underline{K}_{A_m}(\underline{o}_1, \underline{o}_2)$$

Multi-agent subatomic natural deduction

Agent-labelled rules: Agent identity

$$\dfrac{\begin{matrix} [\langle B(\underline{a}_1)A_1\rangle_{a_{k_1}}]^{(1_1)}_{\boldsymbol{a}_i} & [\langle B(\underline{a}_2)A_1\rangle_{a_{k'_1}}]^{(1_2)}_{\boldsymbol{a}_i} & & [\langle B(\underline{a}_1)A_m\rangle_{a_{k_m}}]^{(m_1)}_{\boldsymbol{a}_i} & [\langle B(\underline{a}_2)A_m\rangle_{a_{k'_m}}]^{(m_2)}_{\boldsymbol{a}_i} \\ \boldsymbol{a}_{j_1}\ \dfrac{\mathcal{D}_{1_1}}{B(\underline{a}_2)A_1} & \boldsymbol{a}_{j'_1}\ \dfrac{\mathcal{D}_{1_2}}{B(\underline{a}_1)A_1} & \ldots & \boldsymbol{a}_{j_m}\ \dfrac{\mathcal{D}_{m_1}}{B(\underline{a}_2)A_m} & \boldsymbol{a}_{j'_m}\ \dfrac{\mathcal{D}_{m_2}}{B(\underline{a}_1)A_m} \end{matrix}}{\underline{a}_1 \ddot{\equiv}\ \underline{a}_2}\ \boldsymbol{a}_i\ (\ddot{\equiv}\mathrm{I}),\ 1_1, 1_2, \ldots, m_1, m_2$$

$$\boldsymbol{a}_i\ \dfrac{\boldsymbol{a}_j\ \dfrac{\mathcal{D}_1}{\underline{a}_1 \ddot{\equiv}\ \underline{a}_2} \qquad \boldsymbol{a}_{k_l}\ \dfrac{\mathcal{D}_{l_2}}{B(\underline{a}_1)A_l}}{B(\underline{a}_2)A_l}\ (\ddot{\equiv}\mathrm{E}_l 1) \qquad \boldsymbol{a}_i\ \dfrac{\boldsymbol{a}_j\ \dfrac{\mathcal{D}_1}{\underline{a}_1 \ddot{\equiv}\ \underline{a}_2} \qquad \boldsymbol{a}_{k_l}\ \dfrac{\mathcal{D}_{l_1}}{B(\underline{a}_2)A_l}}{B(\underline{a}_1)A_l}\ (\ddot{\equiv}\mathrm{E}_l 2)$$

where $l \in \{1, \ldots, m\}$

Multi-agent subatomic natural deduction

Canonical derivations

An IBK-derivation which applies an I-rule for a formula in its last step is a **canonical derivation** of that formula.

Theses, theorems, and strictly intuitionistic theorems of IBK-systems

1. Any formula A which can be derived canonically in an IBK-system is a **thesis** of that system.
2. Any thesis A of an IBK-system which can be derived from the empty set of both open (i.e., undischarged) formula assumptions and term assumptions is an IBK-**theorem**.
3. Any theorem of IBK which can be derived exclusively by means of the rules for the standard logical operators (except for ∀I.iii) and intuitionistic absurdity, is a **strictly intuitionistic** IBK-**theorem**.

Multi-agent subatomic natural deduction

Illustration: Belief and knowledge

$$
\boldsymbol{a}_1\ \dfrac{\boldsymbol{a}_1\ \dfrac{\boldsymbol{a}_1\ \dfrac{\boldsymbol{a}_1\ \dfrac{[\langle A\rangle_{\boldsymbol{a}_1}]^{(1)}_{\boldsymbol{a}_1}}{\mathrm{B}_{\underline{a}_1}(A)}\,(\mathrm{B}_{\underline{a}_1}\mathrm{I})}{A \supset \mathrm{B}_{\underline{a}_1}(A)}\,(\supset\mathrm{I}),\ 1 \qquad \boldsymbol{a}_1\ \dfrac{\boldsymbol{a}_1\ \dfrac{[\langle \mathrm{B}_{\underline{a}_1}(A)\rangle_{\boldsymbol{a}_1}]^{(2)}_{\boldsymbol{a}_1}}{A}\,(\mathrm{B}_{\underline{a}_1}\mathrm{E})}{\mathrm{B}_{\underline{a}_1}(A) \supset A}\,(\supset\mathrm{I}),\ 2}{A \leftrightarrow \mathrm{B}_{\underline{a}_1}(A)}\,(\&\mathrm{I})}{\mathrm{K}_{\underline{a}_1}(A \leftrightarrow \mathrm{B}_{\underline{a}_1}(A))}\,(\mathrm{K}_{\underline{a}_1}\mathrm{I}) \tag{8}
$$

Multi-agent subatomic natural deduction

Illustration: Belief and knowledge of theorems

$$
\boldsymbol{a}_1 \; \dfrac{
\boldsymbol{a}_1 \; \dfrac{
\boldsymbol{a}_1 \; \dfrac{
\boldsymbol{a}_1 \; \dfrac{[\langle \mathsf{K}_{\underline{a}_1}(A \supset A)\rangle_{\boldsymbol{a}_1}]^{(1)}_{\boldsymbol{a}_1}}{A \supset A}\,(\mathsf{K}_{\underline{a}_1}\mathsf{E})
}{\mathsf{B}_{\underline{a}_1}(A \supset A)}\,(\mathsf{B}_{\underline{a}_1}\mathsf{I})
}{\mathsf{K}_{\underline{a}_1}(A \supset A) \supset \mathsf{B}_{\underline{a}_1}(A \supset A)}\,(\supset\mathsf{I}),\,1
\quad
\boldsymbol{a}_1 \; \dfrac{
\boldsymbol{a}_1 \; \dfrac{
\boldsymbol{a}_1 \; \dfrac{[\langle A\rangle_{\boldsymbol{a}_1}]^{(2)}_{\boldsymbol{a}_1}}{A \supset A}\,(\supset\mathsf{I}),\,2
}{\mathsf{K}_{\underline{a}_1}(A \supset A)}\,(\mathsf{K}_{\underline{a}_1}\mathsf{I})
}{\mathsf{B}_{\underline{a}_1}(A \supset A) \supset \mathsf{K}_{\underline{a}_1}(A \supset A)}\,(\supset\mathsf{I})
}{\mathsf{K}_{\underline{a}_1}(A \supset A) \leftrightarrow \mathsf{B}_{\underline{a}_1}(A \supset A)}\,(\&\mathsf{I})
\tag{9}
$$

Multi-agent subatomic natural deduction

Illustration: Distributed belief

$C = \mathrm{B}_{\underline{a}_2}(A)\&\mathrm{B}_{\underline{a}_3}(A \supset B)$

$$\boldsymbol{a}_1\ \dfrac{\boldsymbol{a}_1\ \dfrac{\boldsymbol{a}_2\ \dfrac{\boldsymbol{a}_2\ \dfrac{\boldsymbol{a}_2\ \dfrac{\boldsymbol{a}_2\ \dfrac{[\langle C\rangle_{a_1}]^{(1)}_{a_1}}{\mathrm{B}_{\underline{a}_3}(A \supset B)}}{A \supset B} \quad \boldsymbol{a}_2\ \dfrac{\boldsymbol{a}_2\ \dfrac{[\langle C\rangle_{a_1}]^{(1)}_{a_1}}{\mathrm{B}_{\underline{a}_2}(A)}}{A}}{B}}{\mathrm{B}_{\underline{a}_2}(B)} \qquad \boldsymbol{a}_3\ \dfrac{\boldsymbol{a}_3\ \dfrac{\boldsymbol{a}_3\ \dfrac{\boldsymbol{a}_3\ \dfrac{[\langle C\rangle_{a_1}]^{(1)}_{a_1}}{\mathrm{B}_{\underline{a}_3}(A \supset B)}}{A \supset B} \quad \boldsymbol{a}_3\ \dfrac{\boldsymbol{a}_3\ \dfrac{[\langle C\rangle_{a_1}]^{(1)}_{a_1}}{\mathrm{B}_{\underline{a}_2}(A)}}{A}}{B}}{\mathrm{B}_{\underline{a}_3}(B)}}{\mathrm{B}_{\underline{a}_2}(B)\&\mathrm{B}_{\underline{a}_3}(B)}}{(\mathrm{B}_{\underline{a}_2}(A)\&\mathrm{B}_{\underline{a}_3}(A \supset B)) \supset (\mathrm{B}_{\underline{a}_2}(B)\&\mathrm{B}_{\underline{a}_3}(B))}\ (\supset\mathrm{I}),\ 1 \qquad (10)$$

Multi-agent subatomic natural deduction

Illustration: Mutual knowledge

$$
\boldsymbol{a}_1\ \dfrac{
\boldsymbol{a}_1\ \dfrac{
\boldsymbol{a}_1\ \dfrac{\boldsymbol{a}_1\ \dfrac{[\langle \mathrm{K}_{\underline{a}_2}(\mathrm{K}_{\underline{a}_3}(A))\&\mathrm{K}_{\underline{a}_3}(\mathrm{K}_{\underline{a}_2}(A))\rangle_{\boldsymbol{a}_1}]^{(1)}_{\boldsymbol{a}_1}}{\mathrm{K}_{\underline{a}_3}(\mathrm{K}_{\underline{a}_2}(A))}}{\mathrm{K}_{\underline{a}_2}(A)}\ (\mathrm{K}_{\underline{a}_3}\mathrm{E})
\qquad
\boldsymbol{a}_1\ \dfrac{\boldsymbol{a}_1\ \dfrac{[\langle \mathrm{K}_{\underline{a}_2}(\mathrm{K}_{\underline{a}_3}(A))\&\mathrm{K}_{\underline{a}_3}(\mathrm{K}_{\underline{a}_2}(A))\rangle_{\boldsymbol{a}_1}]^{(1)}_{\boldsymbol{a}_1}}{\mathrm{K}_{\underline{a}_2}(\mathrm{K}_{\underline{a}_3}(A))}}{\mathrm{K}_{\underline{a}_3}(A)}\ (\mathrm{K}_{\underline{a}_2}\mathrm{E})
}{\mathrm{K}_{\underline{a}_2}(A)\&\mathrm{K}_{\underline{a}_3}(A)}\ (\&\mathrm{I})
}{(\mathrm{K}_{\underline{a}_2}(\mathrm{K}_{\underline{a}_3}(A))\&\mathrm{K}_{\underline{a}_3}(\mathrm{K}_{\underline{a}_2}(A))) \supset (\mathrm{K}_{\underline{a}_2}(A)\&\mathrm{K}_{\underline{a}_3}(A))}\ (\supset\mathrm{I}),\ 1 \tag{11}
$$

Multi-agent subatomic natural deduction

Illustration: Universal knowledge

$$
\boldsymbol{a}_1\;\cfrac{\boldsymbol{a}_3\;\cfrac{\boldsymbol{a}_1\;\cfrac{\boldsymbol{a}_1\;\cfrac{[\langle \underline{\forall}\ \underline{x}\mathrm{K}_{\underline{x}}(\neg A)\rangle_{\boldsymbol{a}_2}]^{(1)}_{\boldsymbol{a}_1}}{\mathrm{K}_{\underline{a}_1}(\neg A)}\,(\underline{\forall}\mathrm{E})}{\neg A}\,(\mathrm{K}_{\underline{a}_1}\mathrm{E}) \qquad \boldsymbol{a}_2\;\cfrac{[\langle \mathrm{B}_{\underline{a}_2}(A)\rangle_{\boldsymbol{a}_1}]^{(2)}_{\boldsymbol{a}_2}}{A}\,(\mathrm{B}_{\underline{a}_2}\mathrm{E})}{\boldsymbol{a}_2\;\cfrac{\bot}{\mathrm{B}_{\underline{a}_2}(A) \supset \bot}\,(\supset\mathrm{I}),\ 2}\,(\supset\mathrm{E})}{\underline{\forall}\ \underline{x}\mathrm{K}_{\underline{x}}(\neg A) \supset \neg \mathrm{B}_{\underline{a}_2}(A)}\,(\supset\mathrm{I}),\ 1 \qquad (12)
$$

Multi-agent subatomic natural deduction

Illustration: A complex multi-agent belief construction

$$
a_1\ \cfrac{a_4\ \cfrac{a_4\ \cfrac{a_3\ \cfrac{a_3\ \cfrac{a_2\ \cfrac{a_1\ \cfrac{a_1\ \cfrac{a_1\ \cfrac{\varphi^2\Gamma^{a_1}\qquad \alpha_1\Gamma^{a_1}\qquad \alpha_2\Gamma^{a_1}}{\varphi^2\alpha_1\alpha_2}\,(asI)}{\mathrm{B}_{\underline{a}_1}(\varphi^2\alpha_1\alpha_2)}\,(\mathrm{B}_{\underline{a}_1}\mathrm{I})}{\exists x(\mathrm{B}_{\underline{a}_1}(\varphi^2 x\alpha_2))}\,(\exists\mathrm{I})}{\mathrm{B}_{\underline{a}_2}(\exists x(\mathrm{B}_{\underline{a}_1}(\varphi^2 x\alpha_2)))}\,(\mathrm{B}_{\underline{a}_2}\mathrm{I})}{\underline{\exists}\ \underline{x}(\mathrm{B}_{\underline{x}}(\exists x(\mathrm{B}_{\underline{a}_1}(\varphi^2 x\alpha_2))))}\,(\underline{\exists}\mathrm{I})}{\mathrm{B}_{\underline{a}_3}(\underline{\exists}\ \underline{x}(\mathrm{B}_{\underline{x}}(\exists x(\mathrm{B}_{\underline{a}_1}(\varphi^2 x\alpha_2)))))}\,(\mathrm{B}_{\underline{a}_3}\mathrm{I})}{\mathrm{B}_{\underline{a}_4}(\mathrm{B}_{\underline{a}_3}(\underline{\exists}\ \underline{x}(\mathrm{B}_{\underline{x}}(\exists x(\mathrm{B}_{\underline{a}_1}(\varphi^2 x\alpha_2))))))}\,(\mathrm{B}_{\underline{a}_4}\mathrm{I})}{\underline{\forall}\ \underline{y}(\mathrm{B}_{\underline{y}}(\mathrm{B}_{\underline{a}_3}(\underline{\exists}\ \underline{x}(\mathrm{B}_{\underline{x}}(\exists x(\mathrm{B}_{\underline{a}_1}(\varphi^2 x\alpha_2)))))))}\,(\underline{\forall}\mathrm{I})}{\mathrm{B}_{\underline{a}_1}(\underline{\forall}\ \underline{y}(\mathrm{B}_{\underline{y}}(\mathrm{B}_{\underline{a}_3}(\underline{\exists}\ \underline{x}(\mathrm{B}_{\underline{x}}(\exists x(\mathrm{B}_{\underline{a}_1}(\varphi^2 x\alpha_2))))))))}\,(\mathrm{B}_{\underline{a}_1}\mathrm{I}) \tag{13}
$$

Multi-agent subatomic natural deduction

Segments

Let $R \in \{\vee E, \exists E, \underline{\exists} E\}$. A **segment** of length n in a derivation $\mathcal{D}$ in an IBK-system is a sequence $A_1, \ldots, A_n$ of successive occurrences of a formula A in $\mathcal{D}$ such that:

1. for $1 < n$, $i < n$, A_i is a minor premiss of an R-rule application in $\mathcal{D}$ with conclusion A_{i+1};
2. A_n is not a minor premiss of an R-rule application;
3. A_1 is not the conclusion of an R-rule application.

Maximal segments

σ is a **maximal segment** in case A_n is the major premiss of a log^*E-rule of the IBK-system, and either $n > 1$, or $n = 1$ and $A_1 \equiv A_n$ is the conclusion of a log^*I-rule. (A maximum formula is a special case of a maximal segment.)

Multi-agent subatomic natural deduction

A segment (a), (b), (c) of length 3.

$$\boldsymbol{a}_i\ \dfrac{\boldsymbol{a}_k\ \dfrac{\langle \underline{\exists}\ \underline{x}\mathrm{K}_{\underline{x}}(A)\rangle_{\boldsymbol{a}_l} \qquad \boldsymbol{a}_{k'}\ \dfrac{\langle \underline{\exists}\ \underline{x}\mathrm{K}_{\underline{x}}(A)\rangle_{\boldsymbol{a}_{l'}} \qquad \boldsymbol{a}_m\ \dfrac{\begin{matrix}[\langle B\rangle_{\boldsymbol{a}_r}]^{(u)}_{\boldsymbol{a}_m} \\ \boldsymbol{a}_o\ \dfrac{\boldsymbol{a}_q\ \dfrac{\mathcal{D}}{C}}{\mathrm{B}_{\underline{p}}(C)}\ (\mathrm{B}_{\underline{p}}\mathrm{I})\end{matrix}}{\text{(a) } B \supset \mathrm{B}_{\underline{p}}(C)}\ (\supset\mathrm{I}),\ \mathrm{u}}{\text{(b) } B \supset \mathrm{B}_{\underline{p}}(C)}\ (\underline{\exists}\mathrm{E})}{\text{(c) } B \supset \mathrm{B}_{\underline{p}}(C)}\ (\underline{\exists}\mathrm{E}) \qquad \langle B\rangle_{\boldsymbol{a}_j}}{\mathrm{B}_{\underline{p}}(C)}\ (\supset\mathrm{E}) \qquad (14)$$

Multi-agent subatomic natural deduction

Cut rank

The **cut rank** $cr(\mathcal{D})$ of a derivation $\mathcal{D}$ in an IBK-system is a 6-tuple $\langle d, n, e, m, f, o \rangle$, where:

1. d, n, e, m are as above;
2. $f = cr_{log^*}(\mathcal{D})$ where
   a. $cr_{log^*}(\sigma) = |A|$ is the cut rank of a maximal segment σ with formula A;
   b. $cr_{log^*}(\mathcal{D}) = max\{cr_{log^*}(\sigma) \colon \sigma$ is a maximal segment in $\mathcal{D}\}$.

   In case there is no maximal segment, $cr_{log^*}(\mathcal{D}) = 0$.
3. o is the sum of lengths of all critical cuts in $\mathcal{D}$ where a **critical cut** of a derivation $\mathcal{D}$ in IBK is a maximal segment of maximal cut rank from all maximal segments in $\mathcal{D}$.

Derivations in IBK-systems which do not contain (i) maximum atomic sentences, (ii) maximum $\ddot{=}$-sentences, or (iii) critical cuts are **normal**.

Multi-agent subatomic natural deduction

Some detour conversions

$$\dfrac{\boldsymbol{a}_i \; \dfrac{\boldsymbol{a}_j \; \dfrac{\begin{matrix}[\langle A\rangle_{\boldsymbol{a}_l}]^{(u)}_{\boldsymbol{a}_j} \\ \boldsymbol{a}_k \; \dfrac{\mathcal{D}_1}{B}\end{matrix}}{A \supset B}\,(\supset\mathrm{I}),\ u \qquad \boldsymbol{a}_m \; \dfrac{\mathcal{D}_2}{A}}{B}}{}\,(\supset\mathrm{E}) \quad \text{conv} \quad \begin{matrix}\boldsymbol{a}_m \; \dfrac{\mathcal{D}_2}{[A]} \\ \boldsymbol{a}_k \; \dfrac{\mathcal{D}_1}{B}\end{matrix}$$

$$\boldsymbol{a}_i \; \dfrac{\boldsymbol{a}_j \; \dfrac{\boldsymbol{a}_l \; \dfrac{\mathcal{D}_1}{A}}{\mathrm{B}_{\underline{a}_k}(A)}\,(\mathrm{B}_{\underline{a}_k}\mathrm{I})}{A}\,(\mathrm{B}_{\underline{a}_k}\mathrm{E}) \quad \text{conv} \quad \boldsymbol{a}_l \; \dfrac{\mathcal{D}_1}{A}$$

$$\boldsymbol{a}_i \; \dfrac{\boldsymbol{a}_j \; \dfrac{\boldsymbol{a}_k \; \dfrac{\mathcal{D}_1}{\mathrm{E}_{\underline{a}_l}(A)}}{\underline{\exists}\ \underline{x}\mathrm{E}_{\underline{x}}(A)}\,(\underline{\exists}\mathrm{I}) \qquad \begin{matrix}[\langle \mathrm{E}_{\underline{a}_p}(A)\rangle_{\boldsymbol{a}_o}]^{(u)}_{\boldsymbol{a}_i} \\ \boldsymbol{a}_m \; \dfrac{\mathcal{D}_2}{C}\end{matrix}}{C}\,(\underline{\exists}\mathrm{E}),\ u \quad \text{conv} \quad \begin{matrix}\boldsymbol{a}_k \; \dfrac{\mathcal{D}_1}{[\mathrm{E}_{\underline{a}_l}(A)]} \\ \boldsymbol{a}_m \; \dfrac{\mathcal{D}_2}{C}\end{matrix}$$

Multi-agent subatomic natural deduction

Theorem (Normalization)

Any derivation $\mathcal{D}$ in an IBK-system can be transformed into a normal IBK-derivation.

Multi-agent subatomic natural deduction

Units

Let $\mathcal{D}$ be a derivation in an IBK-system.

1. A **unit** in $\mathcal{D}$ is either (i) a segment (a formula being a special case of a segment) or (ii) the occurrence of an $\mathcal{S}^{\ddot{=}}_{\boldsymbol{a}}$-unit in $\mathcal{D}$. We use U, U' (possibly with subscripts) for units.
2. In case U is a term assumption $\tau\Gamma$ in $\mathcal{D}$, τ is **the expression in U**.

Multi-agent subatomic natural deduction

Tracks of IBK-derivations

A **track** of an IBK-derivation $\mathcal{D}$ is a sequence of unit occurrences U_0, ..., U_n such that:

1. U_0 is either a top formula occurrence A_0 in $\mathcal{D}$ not discharged by an application **b** of an R-rule (i.e., $\vee$E, $\exists$E, $\underline{\exists}$E) or U_0 is a top occurrence of an agent-relative term assumption $\tau\Gamma_0^{\boldsymbol{a}}$;
2. U_i is either a formula occurrence A_i for $i < n$ which is not the minor premiss of an instance of an R^*-rule (i.e., $\dddot{=}\mathrm{E}_i$j, $\supset$E, $\underline{\dddot{=}}\mathrm{E}_i$j), and either:
   a. A_i is not the major premiss of an instance of an R-rule and A_{i+1} is directly below A_i, or
   b. A_i is the major premiss of an instance **b** of an R-rule and A_{i+1} is an assumption discharged by **b**; or

   U_i is a term assumption $\tau\Gamma_i^{\boldsymbol{a}}$.
3. U_n is either a formula occurrence A_n which is either:
   a. the minor premiss of an instance of an R^*-rule, or
   b. the conclusion of $\mathcal{D}$, or
   c. the major premiss of an instance **b** of an R-rule in case there is no assumption discharged by **b**; or

   U_n is a term assumption $\tau\Gamma_n^{\boldsymbol{a}}$ which is the conclusion of $\mathcal{D}$.

Multi-agent subatomic natural deduction

Theorem (Structure)

Let $\mathcal{D}$ be a normal IBK-derivation and let π be a track $U_0, \ldots, U_n$ in $\mathcal{D}$. Then there is a segment U_i in π, the minimum part of the track, which divides π into two (possibly empty) parts, an E-part $U_0, \ldots, U_{i-1}$ and an I-part $U_{i+1}, \ldots, U_n$ such that:

1. for each U_j in the **E-part** one has $j < i$, U_j is a (major) premiss of an E-rule, and U_{j+1} is a (stictly positive) subexpression of U_j, and, therefore, each U_j is a (stictly positive) subexpression of U_0;
2. for each U_j in the **I-part** one has $i < j$, and if $j < n$, then U_j is a premiss of an I-rule and a (stictly positive) subexpression of U_{j+1}, so each U_j is a (stictly positive) subexpression of U_n;
3. if $i \neq n$, U_i is also a premiss of an I-rule or of the $\bot$i-rule and a (strictly positive) subexpression of U_0.

Multi-agent subatomic natural deduction

Theorem (Subexpression property)

If $\mathcal{D}$ is a normal IBK-derivation of a unit U from a set of units Γ, then each unit in $\mathcal{D}$ is a subexpression of an expression in $\Gamma \cup \{U\}$.

Multi-agent subatomic natural deduction

Non-normality

$$
\boldsymbol{a}\,\dfrac{\boldsymbol{a}\,\dfrac{\boldsymbol{a}\,\dfrac{\boldsymbol{a}\,\dfrac{\boldsymbol{a}\,\dfrac{[\langle \mathsf{K}_{\underline{a}}(A\supset B)\rangle_{\boldsymbol{a}}]^{(1)}_{\boldsymbol{a}}}{A\supset B}\,(\mathsf{K}_{\underline{a}}\mathrm{E}) \qquad \boldsymbol{a}\,\dfrac{[\langle \mathsf{K}_{\underline{a}}(A)\rangle_{\boldsymbol{a}}]^{(2)}_{\boldsymbol{a}}}{A}\,(\mathsf{K}_{\underline{a}}\mathrm{E})}{B}\,(\supset\mathrm{E})}{\mathsf{K}_{\underline{a}}(B)}\,(\mathsf{K}_{\underline{a}}\mathrm{I})\quad \text{illegal}}{\mathsf{K}_{\underline{a}}(A)\supset \mathsf{K}_{\underline{a}}(B)}\,(\supset\mathrm{I}),\,2}{\mathsf{K}_{\underline{a}}(A\supset B)\supset(\mathsf{K}_{\underline{a}}(A)\supset \mathsf{K}_{\underline{a}}(B))}\,(\supset\mathrm{I}),\,1
$$

Intuitionistic epistemic logic

Comparison with Artemov & Protopopescu [1]

	Principle (where $\Box := K$)	IBK	[1]
1	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	×	√
2	$A \supset \Box A$	×	√
3	$\Box A \supset A$	√	×
4	$\Box A \supset \neg\neg A$	√	√
5	$\Box A \supset \Box\Box A$	×	√
6	$\neg\Box A \supset \Box\neg\Box A$	×	√
7	$\Box\neg A \supset \neg A$	√	√
8	$\Box(A\&B) \supset (\Box A\&\Box B)$	×	√
9	$(\Box A\&\Box B) \supset \Box(A\&B)$	×	√
10	$\Box(A \vee B) \supset (\Box A \vee \Box B)$	×	×
11	$\neg\Box\bot$	√	√
12	$\neg(\Box A\&\neg A)$	√	√
13	$\neg\neg(\Box A \supset A)$	√	√
14	$\neg\Box A \supset \Box\neg A$	×	√
15	$\Box\neg A \supset \neg\Box A$	√	√
16	$\neg\Box A \supset \neg A$	×	√
17	$\neg A \supset \neg\Box A$	√	√
18	$\neg(\neg\Box A\&\neg\Box\neg A)$	×	√

Proof-theoretic semantics

Meaning

The **meaning** of a non-logical constant is given by the term assumption for the constant, and the meaning of a formula is determined by the set of its canonical IBK-derivations.

Intuitionistic intentional identity

Geach's characterization:

"[w]e have **intentional identity** when a number of people, or one person on different occasions, have attitudes with a common focus, whether or not there actually is something at that focus" ([6]: 627).

Geach's example:

Reported outbreak of witch mania in Gotham village:

(1.) Hob thinks a witch has blighted Bob's mare, and Nob wonders whether she (the same witch) killed Cob's sow.

A simplified example:

(2.) Hob believes that Bob's mare is possessed by a demon, and Nob believes that Cob's sow is possessed by it (the same demon) as well.

Intuitionistic intentional identity

Traditional desiderata

A satisfactory analysis of

(2.) Hob *believes* that Bob's mare is possessed by a demon, and Nob *believes* that Cob's sow is possessed by it (the same demon) as well.

has to ensure

- (i) non-existence of demons
- (ii) unspecific reading of 'a demon'
- (iii) cross-attitude convergence of 'a demon' and 'it'

Intuitionistic intentional identity

Model-theoretic proposals

- explain meaning in terms of reference and truth conditions
- use ontology (e.g., individuals, possible worlds, events)
- typically based on classical logic

Present proposal

- explains meaning in terms of IBK-derivations (**proof-theoretic semantics**)
- no ontology used
- based on intuitionistic logic

Intuitionistic intentional identity: First analysis of (2.)

(2.) Hob believes that Bob's mare is possessed by a demon, and Nob believes that Cob's sow is possessed by it (the same demon) as well.

First reading of (2.)

(3.) Bob owns exactly one mare and Hob believes that it is possessed by a demon, and Cob owns exactly one sow and Nob believes that it is possessed by a demon, and all demons are such that if, of any mare which Bob owns, Hob believes that it is possessed by a demon, and of any sow which Cob owns, Nob believes that it is possessed by a demon, then Nob believes that the demons are the same demon.

(Nob forms his belief on the basis of Hob's belief about Bob's mare.)

Intuitionistic intentional identity: First analysis of (2.)

Symbolization of (3.): G

b = 'Bob', c = 'Cob', $\underline{h}$ = 'Hob', $\underline{n}$ = 'Nob', M^1 = 'mare', S^1 = 'sow', D^1 = 'demon', O^2 = 'owns', P^2 = 'is possessed by'.

$G = (A_1 \& A_2) \& A_3$, where:

$$A_1 = \begin{cases} ((\exists x_1(M^1 x_1 \& O^2 b x_1) \\ \& \forall y_1 \forall z_1(((M^1 y_1 \& O^2 b y_1) \& (M^1 z_1 \& O^2 b z_1)) \supset y_1 \ddot{=} z_1)) \\ \& \forall u_1((M^1 u_1 \& O^2 b u_1) \supset \mathrm{B}_{\underline{h}}(\exists v_1(D^1 v_1 \& P^2 u_1 v_1)))) \end{cases}$$

$$A_2 = \begin{cases} ((\exists x_2(S^1 x_2 \& O^2 c x_2) \\ \& \forall y_2 \forall z_2(((S^1 y_2 \& O^2 c y_2) \& (S^1 z_2 \& O^2 c z_2)) \supset y_2 \ddot{=} z_2)) \\ \& \forall u_2((S^1 u_2 \& O^2 c u_2) \supset \mathrm{B}_{\underline{n}}(\exists v_2(D^1 v_2 \& P^2 u_2 v_2)))) \end{cases}$$

$$A_3 = \begin{cases} \forall w_1 \forall w_2[(((D^1 w_1 \& D^1 w_2) \\ \& \forall u_3((M^1 u_3 \& O^2 b u_3) \supset \mathrm{B}_{\underline{h}}(P^2 u_3 w_1))) \\ \& \forall v_3((S^1 v_3 \& O^2 c v_3) \supset \mathrm{B}_{\underline{n}}(P^2 v_3 w_2))) \\ \supset \mathrm{B}_{\underline{n}}(w_1 \ddot{=} w_2)] \end{cases}$$

Digression: Multi-agent subatomic natural deduction

Specific/unspecific uses of ∃I.ii

Let $\exists xA$ be the symbolization of a sentence which admits a specific and an unspecific reading. Convention (cf. [15]):

C-1 When $A(x/\alpha)$ is the premiss of an application of ∃I.ii and $\alpha\Gamma^{\boldsymbol{a}}$ contains no more elements than those which are needed for the derivation of $A(x/\alpha)$ then α, the application of ∃I.ii to $A(x/\alpha)$, and the conclusion $\exists xA$ of this application are called **unspecific**; if $\alpha\Gamma^{\boldsymbol{a}}$ contains more elements than those which are needed for the derivation of $A(x/\alpha)$ then α, the application of ∃I.ii to $A(x/\alpha)$, and the conclusion $\exists xA$ of this application are called **specific**.

Intuitionistic intentional identity: First analysis of (2.)

Derivation of A_{1a}

$$\mathcal{D}_{1a} = \quad \boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{M^1\Gamma^{\boldsymbol{r}} \quad m_1\Gamma^{\boldsymbol{r}}}{M^1 m_1}\,(asI) \qquad \boldsymbol{r}\,\dfrac{O^2\Gamma^{\boldsymbol{r}} \quad b\Gamma^{\boldsymbol{r}} \quad m_1\Gamma^{\boldsymbol{r}}}{O^2 bm_1}\,(asI)}{M^1 m_1 \& O^2 bm_1}}{\underbrace{\exists x_1(M^1 x_1 \& O^2 bx_1)}_{A_{1a}}}\,(\exists I.ii_s) \tag{15}$$

$\exists$I.ii is used in the *specific mode*, since $\{M^1 m_1, O^2 bm_1\} \subset m_1\Gamma^{\boldsymbol{r}}$.

Intuitionistic intentional identity: First analysis of (2.)

Derivation of A_{1b}

$$\mathcal{D}_{1b} = \underbrace{\boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{\mathcal{D}_{1b_{1_1}} \quad \mathcal{D}_{1b_{1_2}} \quad \ldots \quad \mathcal{D}_{1b_{k_1}} \quad \mathcal{D}_{1b_{k_2}}}{m_2 \ddot{=} m_3}\,(\ddot{=}\mathrm{I}),\ 1b_{1_1}, 1b_{1_2}, \ldots, 1b_{k_1}, 1b_{k_2}}{((M^1 m_2 \& O^2 bm_2) \& (M^1 m_3 \& O^2 bm_3)) \supset m_2 \ddot{=} m_3}\,(\supset\mathrm{I}),\ 1b}{\forall z_1(((M^1 m_2 \& O^2 bm_2) \& (M^1 z_1 \& O^2 bz_1)) \supset m_2 \ddot{=} z_1)}\,(\forall\mathrm{I.iii})}{\forall y_1 \forall z_1(((M^1 y_1 \& O^2 by_1) \& (M^1 z_1 \& O^2 bz_1)) \supset y_1 \ddot{=} z_1)}\,(\forall\mathrm{I.iii})}_{A_{1b}} \tag{16}$$

The dots in the application of $\ddot{=}$I indicate that it is used in a *specific* manner. The application of ∀I.iii indicates that subatomic bases matter (and that ∀I.ii cannot be applied here).

Derivation of A_{1b} (contd.)

$$B_{1b} = (M^1 m_2 \;\&\; O^2 bm_2) \;\&(M^1 m_3 \& O^2 bm_3)$$

$$\mathcal{D}_{1b_{1_1}} = \quad \boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{[\langle M^1 m_2\rangle_{\boldsymbol{r}}]_{\boldsymbol{r}}^{(1b_{1_1})}}{M^1\Gamma} \qquad \boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{[\langle B_{1b}\rangle_{\boldsymbol{r}}]_{\boldsymbol{r}}^{(1b)}}{M^1 m_3 \& O^2 bm_3}}{M^1 m_3}}{m_3\Gamma}}{M^1 m_3}\,(asI)$$

$$\mathcal{D}_{1b_{1_2}} = \quad \boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{[\langle M^1 m_3\rangle_{\boldsymbol{r}}]_{\boldsymbol{r}}^{(1b_{1_2})}}{M^1\Gamma} \qquad \boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{[\langle B_{1b}\rangle_{\boldsymbol{r}}]_{\boldsymbol{r}}^{(1b)}}{M^1 m_2 \& O^2 bm_2}}{M^1 m_2}}{m_2\Gamma}}{M^1 m_2}\,(asI)$$

Derivation of A_{1b} (contd.)

$$B_{1b} = (M^1 m_2 \ \& \ O^2 bm_2) \ \& (M^1 m_3 \& O^2 bm_3)$$

$$\mathcal{D}_{1b_{k_1}} = \quad \boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{[\langle O^2 bm_2 \rangle_{\boldsymbol{r}}]_{\boldsymbol{r}}^{(1b_{k_1})}}{O^2 \Gamma} \quad \boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{[\langle B_{1b} \rangle_{\boldsymbol{r}}]_{\boldsymbol{r}}^{(1b)}}{M^1 m_3 \& O^2 bm_3}}{O^2 bm_3}}{b\Gamma} \quad \boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{[\langle B_{1b} \rangle_{\boldsymbol{r}}]_{\boldsymbol{r}}^{(1b)}}{M^1 m_3 \& O^2 bm_3}}{O^2 bm_3}}{m_3 \Gamma}}{O^2 bm_3}\,(asI)$$

$$\mathcal{D}_{1b_{k_2}} = \quad \boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{[\langle O^2 bm_3 \rangle_{\boldsymbol{r}}]_{\boldsymbol{r}}^{(1b_{k_2})}}{O^2 \Gamma} \quad \boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{[\langle B_{1b} \rangle_{\boldsymbol{r}}]_{\boldsymbol{r}}^{(1b)}}{M^1 m_2 \& O^2 bm_2}}{O^2 bm_2}}{b\Gamma} \quad \boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{[\langle B_{1b} \rangle_{\boldsymbol{r}}]_{\boldsymbol{r}}^{(1b)}}{M^1 m_2 \& O^2 bm_2}}{O^2 bm_2}}{m_2 \Gamma}}{O^2 bm_2}\,(asI)$$

Intuitionistic intentional identity: First analysis of (2.)

Derivation of A_{1c}

$$\mathcal{D}_{1c} = \quad \boldsymbol{r}\,\dfrac{\boldsymbol{r}\,\dfrac{\mathcal{D}_{1a}}{A_{1a}} \qquad \boldsymbol{r}\,\dfrac{\mathcal{D}_{1b}}{A_{1b}}}{\underbrace{A_{1a} \& A_{1b}}_{A_{1c}}}\,(\&\mathrm{I}) \tag{17}$$

Derivation of A_{1d}

$$
\mathcal{D}_{1d} = \quad
\boldsymbol{h}\,\dfrac{
\boldsymbol{h}\,\dfrac{
\boldsymbol{h}\,\dfrac{
\boldsymbol{h}\,\dfrac{
\boldsymbol{h}\,\dfrac{
\boldsymbol{h}\,\dfrac{D^1\Gamma^{\boldsymbol{h}} \quad d_1\Gamma^{\boldsymbol{h}}}{D^1 d_1}\,(asI)
\qquad
\boldsymbol{h}\,\dfrac{P^2\Gamma^{\boldsymbol{h}} \quad \boldsymbol{h}\,\dfrac{\boldsymbol{h}\,\dfrac{[\langle M^1 m_4 \& O^2 bm_4\rangle_{\boldsymbol{h}}]^{(1c)}_{\boldsymbol{h}}}{M^1 m_4}}{m_4\Gamma^{\boldsymbol{h}}} \quad d_1\Gamma^{\boldsymbol{h}}}{P^2 m_4 d_1}\,(asI)
}{D^1 d_1 \& P^2 m_4 d_1}
}{\exists v_1(D^1 v_1 \& P^2 m_4 v_1)}\,(\exists\mathrm{I.ii}_u)
}{\mathrm{B}_{\underline{h}}(\exists v_1(D^1 v_1 \& P^2 m_4 v_1))}\,(\mathrm{B}_{\underline{h}}\mathrm{I})
}{(M^1 m_4 \& O^2 bm_4) \supset \mathrm{B}_{\underline{h}}(\exists v_1(D^1 v_1 \& P^2 m_4 v_1))}\,(\supset\mathrm{I}),\ 1c
}{\underbrace{\forall u_1((M^1 u_1 \& O^2 bu_1) \supset \mathrm{B}_{\underline{h}}(\exists v_1(D^1 v_1 \& P^2 u_1 v_1)))}_{A_{1d}}}\,(\forall\mathrm{I.iii})
\tag{18}
$$

$\exists$I.ii is used in the *unspecific mode*, since $d_1\Gamma^{\boldsymbol{h}} = \{D^1 d_1, P^2 m_4 d_1\}$.

Intuitionistic intentional identity: First analysis of (2.)

Derivation of A_1

$$\mathcal{D}_1 = \dfrac{\boldsymbol{r}\,\dfrac{\mathcal{D}_{1c}}{A_{1c}} \quad \boldsymbol{h}\,\dfrac{\mathcal{D}_{1d}}{A_{1d}}}{\boldsymbol{r}\,\underbrace{A_{1c} \& A_{1d}}_{=A_1}}\,(\&\mathrm{I}) \tag{19}$$

The reporter concludes A_1 on the basis of his own and Hob's conclusions.

Intuitionistic intentional identity: First analysis of (2.)

A_2

A_2 is a conjunction of A_{2c} and A_{2d}, where:

$$A_{2c} = \underbrace{\exists x_2(S^1x_2 \& O^2cx_2)}_{A_{2a}} \& \underbrace{\forall y_2 \forall z_2(((S^1y_2 \& O^2cy_2) \& (S^1z_2 \& O^2cz_2)) \supset y_2 \ddot{=} z_2)}_{A_{2b}}$$

$$A_{2d} = \forall u_2((S^1u_2 \& O^2cu_2) \supset \mathrm{B}_{\underline{n}}(\exists v_2(D^1v_2 \& P^2u_2v_2)))$$

The sentences symbolized are:

A_{2a}: There is at least one sow which Cob owns.

A_{2b}: All the sows Cob owns are the same sow.

A_{2c}: Cob owns exactly one sow.

A_{2d}: If something is a sow which Cob owns, then Nob believes that it is possessed by a demon.

A_2: Cob owns exactly one sow and Nob believes that it is possessed by a demon.

The derivation of A_2 is exactly analogous to that of A_1.

Intuitionistic intentional identity: First analysis of (2.)

Derivation of A_3

$B_1 = D^1 d_3 \& D^1 d_4$
$B_2(d_3) = \forall u_3((M^1 u_3 \& O^2 b u_3) \supset \mathrm{B}_{\underline{h}}(P^2 u_3 d_3))$
$B_3(d_4) = \forall v_3((S^1 v_3 \& O^2 c v_3) \supset \mathrm{B}_{\underline{n}}(P^2 v_3 d_4))$ Let $k' < k$ and $k' = 3$.

$$
\underbrace{\boldsymbol{n}\,\frac{\boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{\mathcal{D}_{3_{1_1}}, \mathcal{D}_{3_{1_2}}, \mathcal{D}_{3_{2_1}}, \mathcal{D}_{3_{2_2}}, \mathcal{D}_{3_{k'_1}}, \mathcal{D}_{3_{k'_2}}}{d_3 \mathrel{\ddot{=}} d_4}\,(\ddot{=}\mathrm{I}),\ 3_{1_1}, 3_{1_2}, 3_{2_1}, 3_{2_2}, 3_{k'_1}, 3_{k'_2}}{\mathrm{B}_{\underline{n}}(d_3 \mathrel{\ddot{=}} d_4)}}{(((D^1 d_3 \& D^1 d_4) \& B_2(d_3)) \& B_3(d_4)) \supset \mathrm{B}_{\underline{n}}(d_3 \mathrel{\ddot{=}} d_4)}\,(\supset\mathrm{I}),\ 3}{\forall w_2[(((D^1 d_3 \& D^1 w_2) \& B_2(d_3)) \& B_3(w_2)) \supset \mathrm{B}_{\underline{n}}(d_3 \mathrel{\ddot{=}} w_2)]}\,(\forall\mathrm{I.iii})}{\forall w_1 \forall w_2[(((D^1 w_1 \& D^1 w_2) \& B_2(w_1)) \& B_3(w_2)) \supset \mathrm{B}_{\underline{n}}(w_1 \mathrel{\ddot{=}} w_2)]}\,(\forall\mathrm{I.iii})}_{A_3} \qquad (16)
$$

The lack of dots in the application of $\ddot{=}$I indicates that it used in an *unspecific* manner.

A_3 makes use of belief *de nomine* (there is no *res* or individual at the focus).

Intuitionistic intentional identity: First analysis of (2.)

Derivation of A_3 (contd.)

$$B_1 = D^1 d_3 \& D^1 d_4$$
$$B_2(d_3) = \forall u_3((M^1 u_3 \& O^2 b u_3) \supset B_{\underline{h}}(P^2 u_3 d_3))$$
$$B_3(d_4) = \forall v_3((S^1 v_3 \& O^2 c v_3) \supset B_{\underline{n}}(P^2 v_3 d_4))$$

$$\mathcal{D}_{3_{1_1}} = \quad \text{(a)} \quad \boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{[\langle D^1 d_3\rangle_{\boldsymbol{n}}]_{\boldsymbol{n}}^{(3_{1_1})}}{D^1 \Gamma^{\boldsymbol{n}}} \qquad d_4 \Gamma^{\boldsymbol{n}}}{D^1 d_4}$$

$$\mathcal{D}_{3_{1_2}} = \quad \text{(b)} \quad \boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{[\langle D^1 d_4\rangle_{\boldsymbol{n}}]_{\boldsymbol{n}}^{(3_{1_2})}}{D^1 \Gamma^{\boldsymbol{n}}} \qquad \boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{[\langle (B_1 \& B_2(d_3)) \& B_3(d_4)\rangle_{\boldsymbol{n}}]_{\boldsymbol{n}}^{(3)}}{B_1 \& B_2(d_3)}}{B_1}}{D^1 d_3}}{d_3 \Gamma^{\boldsymbol{n}}}}{D^1 d_3} \qquad (11)$$

Derivation of A_3 (contd.)

$$B_1 = D^1 d_3 \& D^1 d_4$$
$$B_2(d_3) = \forall u_3((M^1 u_3 \& O^2 b u_3) \supset B_{\underline{h}}(P^2 u_3 d_3))$$
$$B_3(d_4) = \forall v_3((S^1 v_3 \& O^2 c v_3) \supset B_{\underline{n}}(P^2 v_3 d_4))$$

$$\mathcal{D}_{3_{2_1}} = \quad n\,\dfrac{n\,\dfrac{[\langle P^2 s_5 d_3\rangle_n]_n^{(3_{2_1})}}{P^2\Gamma^n} \qquad n\,\dfrac{[\langle P^2 s_5 d_3\rangle_n]_n^{(3_{2_1})}}{s_5\Gamma^n} \qquad n\,\dfrac{n\,\dfrac{n\,\dfrac{n\,\dfrac{[\langle (B_1 \& B_2(d_3)) \& B_3(d_4)\rangle_n]_n^{(3)}}{B_3(d_4)}}{(S^1 s_5 \& O^2 c s_5) \supset \mathrm{B}_{\underline{n}}(P^2 s_5 d_4)} \qquad n\,\dfrac{\mathcal{D}_{3a}}{B_{3a}}}{\mathrm{B}_{\underline{n}}(P^2 s_5 d_4)}}{P^2 s_5 d_4}}{d_4\Gamma^n}}{P^2 s_5 d_4} \qquad (12)$$

$$\text{where } \mathcal{D}_{3a} = \quad n\,\dfrac{n\,\dfrac{S^1\Gamma^n \qquad s_5\Gamma^n}{S^1 s_5} \qquad n\,\dfrac{O^2\Gamma^n \qquad c\Gamma^n \qquad s_5\Gamma^n}{O^2 c s_5}}{\underbrace{S^1 s_5 \& O^2 c s_5}_{B_{3a}}}$$

Derivation of A_3 (contd.)

$$B_1 = D^1 d_3 \& D^1 d_4$$
$$B_2(d_3) = \forall u_3((M^1 u_3 \& O^2 b u_3) \supset B_{\underline{h}}(P^2 u_3 d_3))$$
$$B_3(d_4) = \forall v_3((S^1 v_3 \& O^2 c v_3) \supset B_{\underline{n}}(P^2 v_3 d_4))$$

$$\mathcal{D}_{3_{2_2}} = \quad n\,\frac{n\,\dfrac{[\langle {\color{green}P^2 s_5 d_4}\rangle_{\boldsymbol{n}}]_{\boldsymbol{n}}^{(3_{2_2})}}{P^2\Gamma^{\boldsymbol{n}}} \qquad n\,\dfrac{[\langle {\color{green}P^2 s_5 d_4}\rangle_{\boldsymbol{n}}]_{\boldsymbol{n}}^{(3_{2_2})}}{s_5\Gamma^{\boldsymbol{n}}} \qquad n\,\dfrac{n\,\dfrac{n\,\dfrac{n\,\dfrac{n\,\dfrac{[\langle (B_1 \& B_2(d_3)) \& B_3(d_4)\rangle_{\boldsymbol{n}}]_{\boldsymbol{n}}^{(3)}}{B_1 \& B_2(d_3)}}{B_2(d_3)}}{(M^1 m_5 \& O^2 b m_5) \supset \mathrm{B}_{\underline{h}}(P^2 m_5 d_3)} \qquad n\,\dfrac{\mathcal{D}_{3b}}{B_{3b}}}{\mathrm{B}_{\underline{h}}(P^2 m_5 d_3)}}{P^2 m_5 d_3}}{d_3\Gamma^{\boldsymbol{n}}}}{{\color{green}P^2 s_5 d_3}} \tag{13}$$

$$\text{where } \mathcal{D}_{3b} = \quad n\,\frac{n\,\dfrac{M^1\Gamma^{\boldsymbol{n}} \qquad m_5\Gamma^{\boldsymbol{n}}}{M^1 m_5} \qquad n\,\dfrac{O^2\Gamma^{\boldsymbol{n}} \qquad b\Gamma^{\boldsymbol{n}} \qquad m_5\Gamma^{\boldsymbol{n}}}{O^2 b m_5}}{\underbrace{M^1 m_5 \& O^2 b m_5}_{B_{3b}}}$$

Derivation of A_3 (contd.)

$B_1 = D^1 d_3 \& D^1 d_4$

$B_2(d_3) = \forall u_3((M^1 u_3 \& O^2 b u_3) \supset B_{\underline{h}}(P^2 u_3 d_3))$

$B_3(d_4) = \forall v_3((S^1 v_3 \& O^2 c v_3) \supset B_{\underline{n}}(P^2 v_3 d_4))$ *Let $k' < k$ and $k' = 3$.*

$$\mathcal{D}_{3_{k'_1}} = \boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{[\langle {P^2 m_5 d_3}\rangle_{\boldsymbol{n}}]_{\boldsymbol{n}}^{(3_{k'_1})}}{P^2\Gamma^{\boldsymbol{n}}} \qquad \boldsymbol{n}\,\dfrac{[\langle {P^2 m_5 d_3}\rangle_{\boldsymbol{n}}]_{\boldsymbol{n}}^{(3_{k'_1})}}{m_5\Gamma^{\boldsymbol{n}}} \qquad \boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{[\langle((B_1 \& B_2(d_3)) \& B_3(d_4))\rangle_{\boldsymbol{n}}]_{\boldsymbol{n}}^{(3)}}{B_3(d_4)}}{(S^1 s_5 \& O^2 c s_5) \supset \mathrm{B}_{\underline{n}}(P^2 s_5 d_4)} \qquad \boldsymbol{n}\,\dfrac{\mathcal{D}_{3a}}{B_{3a}}}{\mathrm{B}_{\underline{n}}(P^2 s_5 d_4)}}{P^2 s_5 d_4}}{d_4\Gamma^{\boldsymbol{n}}}}{P^2 m_5 d_4} \qquad (14)$$

Derivation of A_3 (contd.)

$$B_1 = D^1 d_3 \& D^1 d_4$$
$$B_2(d_3) = \forall u_3((M^1 u_3 \& O^2 bu_3) \supset B_{\underline{h}}(P^2 u_3 d_3))$$
$$B_3(d_4) = \forall v_3((S^1 v_3 \& O^2 cv_3) \supset B_{\underline{n}}(P^2 v_3 d_4))$$

Let $k' < k$ and $k' = 3$.

$$\mathcal{D}3_{k'_2} = \frac{ n\,\dfrac{[\langle P^2 m_5 d_4\rangle_{\boldsymbol{n}}]_{\boldsymbol{n}}^{(3_{k'_2})}}{P^2\Gamma^{\boldsymbol{n}}} \qquad n\,\dfrac{[\langle P^2 m_5 d_4\rangle_{\boldsymbol{n}}]_{\boldsymbol{n}}^{(3_{k'_2})}}{m_5\Gamma^{\boldsymbol{n}}} \qquad n\,\dfrac{n\,\dfrac{n\,\dfrac{n\,\dfrac{n\,\dfrac{[\langle (B_1 \& B_2(d_3)) \& B_3(d_4)\rangle_{\boldsymbol{n}}]_{\boldsymbol{n}}^{(3)}}{B_1 \& B_2(d_3)}}{B_2(d_3)}}{(M^1 m_5 \& O^2 bm_5) \supset \mathrm{B}_{\underline{h}}(P^2 m_5 d_3)} \qquad n\,\dfrac{\mathcal{D}_{3b}}{B_{3b}}}{\mathrm{B}_{\underline{h}}(P^2 m_5 d_3)}}{P^2 m_5 d_3}}{d_3\Gamma^{\boldsymbol{n}}} }{ P^2 m_5 d_3 } \tag{15}$$

Intuitionistic intentional identity: First analysis of (2.)

Derivation of G

$$\boldsymbol{r}\,\cfrac{\boldsymbol{r}\,\cfrac{\boldsymbol{r}\,\cfrac{\mathcal{D}_1}{A_1} \quad \boldsymbol{r}\,\cfrac{\mathcal{D}_2}{A_2}}{A_1 \& A_2}\,(\&\mathrm{I}) \qquad \boldsymbol{n}\,\cfrac{\mathcal{D}_3}{A_3}}{\underbrace{(A_1 \& A_2) \& A_3}_{G}}\,(\&\mathrm{I}) \tag{17}$$

Meaning of G

The meaning of G is determined by the set of its (possibly non-normal) canonical IBK-derivations. (17) is a member of this set.

Intuitionistic intentional identity: Second analysis of (2.)

Second reading of (2.)

(4.) There is a demon of which Hob believes that Bob's mare is possessed by it and of which Nob believes that Cob's sow is possessed by it.

(Permits that Nob need not have any beliefs concerning Hob or concerning Bob's mare, and that Hob need not have any beliefs about Nob or about Cob's sow.)

Symbolization of (4.): G'

$$
\begin{aligned}
G' = \exists w(& \\
& [A_{1c} \& \mathrm{B}_{\underline{h}}(D^1 w \& (\forall u_1((M^1 u_1 \& O^2 b u_1) \supset P^2 u_1 w)))] \\
& \& \\
& [A_{2c} \& \mathrm{B}_{\underline{n}}(D^1 w \& (\forall u_2((S^1 u_2 \& O^2 c u_2) \supset P^2 u_2 w)))])
\end{aligned}
$$

Intuitionistic intentional identity: Second analysis of (2.)

Derivation of G'

$A_{1e}(d) = \mathrm{B}_{\underline{h}}(D^1 d \& (\forall u_1((M^1 u_1 \& O^2 b u_1) \supset P^2 u_1 d)))$
$A_{2e}(d) = \mathrm{B}_{\underline{n}}(D^1 d \& (\forall u_2((S^1 u_2 \& O^2 c u_2) \supset P^2 u_2 d)))$

$$\boldsymbol{r}\,\cfrac{\boldsymbol{r}\,\cfrac{\boldsymbol{r}\,\cfrac{\boldsymbol{r}\,\cfrac{\mathcal{D}_{1c}}{A_{1c}} \quad \boldsymbol{h}\,\cfrac{\mathcal{D}_{1e}}{A_{1e}(d)}}{A_{1c} \& A_{1e}(d)} \qquad \boldsymbol{r}\,\cfrac{\boldsymbol{r}\,\cfrac{\mathcal{D}_{2c}}{A_{2c}} \quad \boldsymbol{n}\,\cfrac{\mathcal{D}_{2e}}{A_{2e}(d)}}{A_{2c} \& A_{2e}(d)}}{[A_{1c} \& A_{1e}(d)] \& [A_{2c} \& A_{2e}(d)]}}{\exists w([A_{1c} \& A_{1e}(w)] \& [A_{2c} \& A_{2e}(w)])}\,(\exists \mathrm{I}_u) \qquad (20)$$

Derivation of $A_{1e}(d)$

$$\mathcal{D}_{1e} = \quad h\,\dfrac{h\,\dfrac{h\,\dfrac{D^1\Gamma^{\boldsymbol{h}} \qquad d\Gamma^{\boldsymbol{h}}}{D^1 d}\,(asI) \qquad h\,\dfrac{h\,\dfrac{h\,\dfrac{P^2\Gamma^{\boldsymbol{h}} \qquad h\,\dfrac{h\,\dfrac{[\langle M^1 m \& O^2 bm\rangle_{\boldsymbol{h}}]_{\boldsymbol{h}}^{(1e)}}{O^2 bm}\,(\&\mathrm{E}2)}{m\Gamma^{\boldsymbol{h}}}\,(as\mathrm{E}_2) \qquad d\Gamma^{\boldsymbol{h}}}{P^2 md}\,(asI)}{(M^1 m \& O^2 bm) \supset P^2 md}\,(\supset \mathrm{I}),\ 1e}{\forall u_1((M^1 u_1 \& O^2 bu_1) \supset P^2 u_1 d)}\,(\forall \mathrm{I.iii})}{D^1 d \& (\forall u_1((M^1 u_1 \& O^2 bu_1) \supset P^2 u_1 d))}\,(\&\mathrm{I})}{\underbrace{\mathrm{B}_{\underline{h}}(D^1 d \& (\forall u_1((M^1 u_1 \& O^2 bu_1) \supset P^2 u_1 d)))}_{A_{1e}(d)}}\,(\mathrm{B}_{\underline{h}}\mathrm{I}) \qquad (18)$$

$$d\Gamma^{\boldsymbol{h}} = \{D^1 d, P^2 md\}$$

Intuitionistic intentional identity: Second analysis of (2.)

Derivation of $A_{2e}(d)$

$$
\mathcal{D}_{2e} = \quad \boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{D^1\Gamma^{\boldsymbol{n}} \qquad d\Gamma^{\boldsymbol{n}}}{D^1 d}\,(as\mathrm{I}) \qquad \boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{P^2\Gamma^{\boldsymbol{n}} \qquad \boldsymbol{n}\,\dfrac{\boldsymbol{n}\,\dfrac{[\langle S^1 s \& O^2 cs\rangle_{\boldsymbol{n}}]^{(2e)}_{\boldsymbol{n}}}{O^2 cs}\,(\&\mathrm{E}2)}{s\Gamma^{\boldsymbol{n}}}\,(as\mathrm{E}_2) \qquad d\Gamma^{\boldsymbol{n}}}{P^2 sd}\,(as\mathrm{I})}{(S^1 s \& O^2 cs) \supset P^2 sd}\,(\supset\mathrm{I}),\ 2e}{\forall u_2((S^1 u_2 \& O^2 cu_2) \supset P^2 u_2 d)}\,(\forall\mathrm{I.iii})}{D^1 d \& (\forall u_2((S^1 u_2 \& O^2 cu_2) \supset P^2 u_2 d))}\,(\&\mathrm{I})}{\mathrm{B}_{\underline{n}}(D^1 d \& (\forall u_2((S^1 u_2 \& O^2 cu_2) \supset P^2 u_2 d)))}\,(\mathrm{B}_{\underline{n}}\mathrm{I}) \qquad (19)
$$

$$
d\Gamma^{\boldsymbol{n}} = \{D^1 d, P^2 sd\}
$$

Intuitionistic intentional identity: Second analysis of (2.)

Derivation of G': Ignorance

Hob's ignorance with respect to Nob or Cob's sow, and Nob's ignorance with respect to Hob or Bob's mare, can be guaranteed by the following stipulations concerning the multi-agent belief base of the IBK-system:

1. $n\Gamma^{\boldsymbol{h}} = c\Gamma^{\boldsymbol{h}} = s\Gamma^{\boldsymbol{h}} = \varnothing$; $S^1 s \notin S^1\Gamma^{\boldsymbol{h}}$; $O^2 cs \notin O^2\Gamma^{\boldsymbol{h}}$; $P^2 sd \notin P^2\Gamma^{\boldsymbol{h}}$.
2. $h\Gamma^{\boldsymbol{n}} = b\Gamma^{\boldsymbol{n}} = m\Gamma^{\boldsymbol{n}} = \varnothing$; $M^1 m \notin M^1\Gamma^{\boldsymbol{n}}$; $O^2 bm \notin O^2\Gamma^{\boldsymbol{n}}$; $P^2 md \notin P^2\Gamma^{\boldsymbol{n}}$.
3. Recall: $d\Gamma^{\boldsymbol{h}} = \{D^1 d, P^2 md\}$ and $d\Gamma^{\boldsymbol{n}} = \{D^1 d, P^2 sd\}$.

Derivation of G': Common focus

Illustration: We assume Hob and Nob base their beliefs on the *Gotham News* report that there is a demon called 'Don'. Let $\boldsymbol{g}$ be the agent-label for 'Gotham News' and let $d =$ 'Don': $d\Gamma^{\boldsymbol{g}} = d\Gamma^{\boldsymbol{h}} \cap d\Gamma^{\boldsymbol{n}} = \{D^1 d\}$. Intuitively, $d\Gamma^{\boldsymbol{g}}$ is the common focus of Hob's and Nob's attitudes.

Intuitionistic multi-agent subatomic natural deduction

Conclusion

IBK-systems are:

- **autarkical**
  (not extracted from model-theoretic possible worlds truth conditions)
- **Gentzenian**
  (admit a proof-theoretic semantics due to normalization)
- **fully analytic**
  (due to subexpression/subformula property)
- **intuitionistic**
  (support a constructive conception of truth, meaning, and belief)
- **versatile**
  (suitable for, e.g., de dicto/de nomine/de se, mutual, distributed, universal belief/knowledge; intentional identity)

Tack så mycket!