Propositional Attitude Operators via Homotopy Type Theory

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Frege's Puzzle about Propositional Attitudes (1892)

- In simple cases, substitution of equals for equals works in natural language:

$$\frac{\text{The author of Waverley is Scottish.}\quad \text{Scott is the author of Waverley.}}{\text{Scott is Scottish.}}$$

- But not in more complex cases such as "propositional attitudes":

$$\frac{\text{George IV knows that the author of Waverley is Scottish.}\quad \text{Scott is the author of Waverley.}}{\text{George IV knows that Scott is Scottish.}}\ ✗$$

- Conclusion: though "Scott" is equal to "the author of Waverley" in a way, they are unequal in another way.

Logics with 2 Equalities

- It seems we want to say that “Scott” and “the author of Waverley” are extensionally equal but intensionally unequal.
- ...and to model this analysis in a logic!
- Inspired by Frege, several logics integrate 2 equality predicates, incl. IMLTT/HoTT (1986/2013), Fox and Lappin (2005).

Fox and Lappin *vs.* HoTT

- In HoTT, all predicates respect extensional equality. IOW, we can't model:

George IV knows that the author of Waverley is Scottish.

$$\frac{\text{Scott is the author of Waverley.}}{\text{George IV knows that Scott is Scottish.}} ✗$$

- In Fox and Lappin, no predicates respect extensional equality, as far as the deductive system is concerned. IOW, we can't model:

$$\frac{\begin{array}{c}\text{The author of Waverley is Scottish.}\\ \text{Scott is the author of Waverley.}\end{array}}{\text{Scott is Scottish.}}$$

- Note: we can't simply stipulate that some predicates are intensional and some extensional: "is Scottish" is extensional or intensional depending on whether it is in a propositional attitude context or not!

An Answer: HoTT + Montague's IL

- It seems we need some resources in the logic for distinguishing intensional *v.s.* extensional contexts, predicates, *etc.*
- We turn to another Frege-inspired system: Montague's intensional logic.
- The present approach combines HoTT and IL in a fairly natural way.

Homotopy Type Theory

- Homotopy Type Theory (HoTT) is a foundation for mathematics (i.e. an alternative to ZFC set theory) developed over the last ~ 10 years (see UFP 2013)
- A kind of (intensional Martin-Löf dependent) type theory, augmented with geometrically-motivated axioms
- More amenable to computer-checking of proofs than set theory (simpler/more direct encodings for mathematical structures)
- *E.g.* the present logic is implemented in Agda (Vezzosi 2019)

Equality in HoTT

Central to HoTT is its subtle, 'intensional' treatment of equality.
There are 2 basic notions of equality:

- ($\equiv$): 'Judgemental equality', corresponding (roughly) to equality of mathematical objects, which we think of as equality of intension
- ($=$): 'Typal equality', corresponding to isomorphism/equivalence (including logical equivalence of propositions), which we think of as equality of extension

Of course $t \equiv u$ implies $t = u$, but not vice versa!

Remark:

Propositions in HoTT (+ classical logic) form a Boolean pre-algebra (AKA Boolean pre-lattice, c.f. Fox *et al.* 2002, Fox and Lappin 2005)

Combining HoTT and Montague

Objective:

Combine HoTT and Montague to interpret propositional attitude operators.

Proposal:

Comonadic Homotopy Type Theory (**CHoTT**)

CHoTT (History)

- Line of work including Nanevski et al. (2007), Shulman (2018)
  - Work connecting Montague with comonads: Awodey et al. (2015, 2016), Zwanziger (2017)
- **CHoTT** is a fragment of Shulman (2018), chosen to be compatible with the hyperintensional application

CHoTT (Variable Judgements)

CHoTT has two variable judgements,

$$u :: A$$

and

$$x : A$$

We will say (at variance with prior terminology) that "u is an intensional variable of type A," when $u :: A$ and that "x is an extensional variable of type A," when $x : A$. A term in an intensional variable will not be required to respect extensional equality with respect to that variable.

CHoTT (Hypothetical Judgements)

The hypothetical judgements of **CHoTT** have the form

$$\Delta \mid \Gamma \vdash t : B$$

and

$$\Delta \mid \Gamma \vdash t \equiv u : B$$

CHoTT (Variable and Context Rules)

$$\frac{}{\cdot \mid \cdot \ \mathsf{ctx}} \text{ ctx-Emp.}$$

$$\frac{\Delta \mid \Gamma \vdash B : U}{\Delta \mid \Gamma, x : B \ \mathsf{ctx}} \text{ ctx-Ext.}^e \qquad \frac{\Delta \mid \Gamma, x : A, \Gamma' \ \mathsf{ctx}}{\Delta \mid \Gamma, x : A, \Gamma' \vdash x : A} \text{ Var.}^e$$

$$\frac{\Delta \mid \cdot \vdash B : U}{\Delta, u :: B \mid \cdot \ \mathsf{ctx}} \text{ ctx-Ext.}^i \qquad \frac{\Delta, u :: A, \Delta' \mid \Gamma \ \mathsf{ctx}}{\Delta, u :: A, \Delta' \mid \Gamma \vdash u : A} \text{ Var.}^i$$

Figure: The Extensional ($-^e$) and Intensional ($-^i$) Context Rules

CHoTT (The HoTT Part)

- We import the usual homotopy type theoretical notions (as found in UFP *op. cit.*), including $\prod$- and $\sum$-types (corresponding to the quantifiers $\forall$ and $\exists$), universe polymorphism, $=$-types, higher inductive types (HITs), and univalence.
- However, to keep things simple, the typing rules are assumed to manipulate *extensional context variables only*. E.g. $\lambda u :: A.u :: A$ is ill-typed.

CHoTT (The HoTT Part)

Identity Types

The restriction of the rules for $=$-types to extensional variables is crucial, though. It ensures that we have the principle

$$\frac{\Delta \mid \Gamma, x : A \vdash B : U \qquad \begin{array}{c} \Delta \mid \Gamma \vdash s, t : A \\ \Delta \mid \Gamma \vdash p : s =_A t \\ \Delta \mid \Gamma \vdash q : B[s/x] \end{array}}{\Delta \mid \Gamma \vdash \ell(s,t,p,q) : B[t/x]} \text{ Indiscernibility}^e$$

in which the variable x of the predicate B is extensional, but *not* the principle

$$\frac{\Delta, u :: A \mid \cdot \vdash B : U \qquad \begin{array}{c} \Delta \mid \cdot \vdash s, t : A \\ \Delta \mid \cdot \vdash p : s =_A t \\ \Delta \mid \cdot \vdash q : B[s/x] \end{array}}{\Delta \mid \Gamma \vdash \ell(s,t,p,q) : B[t/x]} \text{ Indiscernibility}^i$$

in which the variable u of the predicate B is intensional.

CHoTT (♭-types)

$$\frac{\Delta \mid \cdot \vdash B : U}{\Delta \mid \Gamma \vdash \flat B : U}\ \flat\text{-Form.} \qquad \frac{\Delta \mid \cdot \vdash t : B}{\Delta \mid \Gamma \vdash t^\flat : \flat B}\ \flat\text{-Intro.}$$

$$\frac{\Delta \mid \Gamma, x : \flat A \vdash B : U \qquad \begin{array}{c}\Delta \mid \Gamma \vdash s : \flat A \\ \Delta, u :: A \mid \Gamma \vdash t : B[u^\flat/x]\end{array}}{\Delta \mid \Gamma \vdash (\text{let } u^\flat := s \text{ in } t) : B[s/x]}\ \flat\text{-Elim.}$$

$$\frac{\Delta \mid \Gamma, x : \flat A \vdash B : U \qquad \begin{array}{c}\Delta \mid \cdot \vdash s : A \\ \Delta, u :: A \mid \Gamma \vdash t : B[u^\flat/x]\end{array}}{\Delta \mid \Gamma \vdash \text{let } u^\flat := s^\flat \text{ in } t \equiv t[s/u] : B[s^\flat/x]}\ \flat\text{-}\beta\text{-Conversion}$$

Figure: The Rules for ♭

Simply Typed Elim. for $\flat$

We can derive an elim. rule corresponding to Montague's extension operator. This rule,

$$\frac{\Delta \mid \cdot \vdash B : U \qquad \Delta \mid \Gamma \vdash t : \flat B}{\Delta \mid \Gamma \vdash t_\flat : B}\ \flat\text{-Elim.-Simple}$$

is derived by

$$\frac{\dfrac{\Delta \mid \cdot \vdash B : U}{\Delta \mid \Gamma, x : \flat B \vdash B : U} \qquad \Delta \mid \Gamma \vdash t : \flat B \qquad \dfrac{}{\Delta, u :: B \mid \Gamma \vdash u : B}}{\Delta \mid \Gamma \vdash (\text{let } u^\flat := t \text{ in } u) \equiv t_\flat : B}$$

Example

Scott *vs.* the Author of Waverley

- $g, s, a : E$ translate "George IV", "Scott", "the author of Waverley"
- $S : E \to U$ translates "is Scottish"
- $K : E \to \flat U \to U$ translates "knows"

Example (continued)

Scott *vs.* the Author of Waverley

- $s =_E a$ translates "Scott is the author of Waverley"
- $K(g, S(a)^\flat)$ translates "George IV knows that the author of Waverley is Scottish"
- $K(g, S(s)^\flat)$ translates "George IV knows that Scott is Scottish"

Example (continued)

Scott *vs.* the Author of Waverley

But we do not have

$$\frac{K(g, S(a)^\flat) \qquad s =_E a}{K(g, S(s)^\flat)}$$

since this would involve Indiscernibility of Identicals^i!

Future Work

- Semantics of Montague's intension operator as quotation?
- Dedicated syntax for quotation operators?

Thanks!